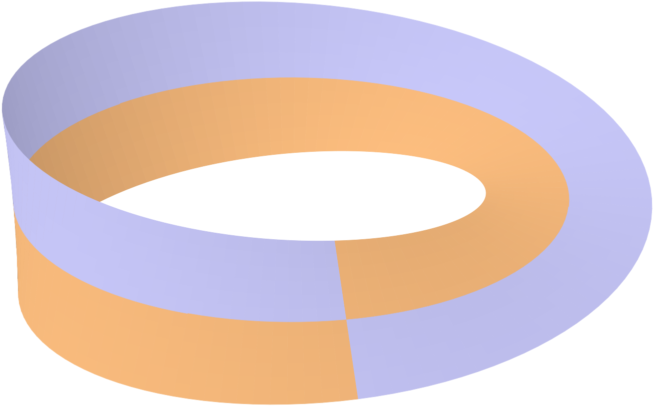
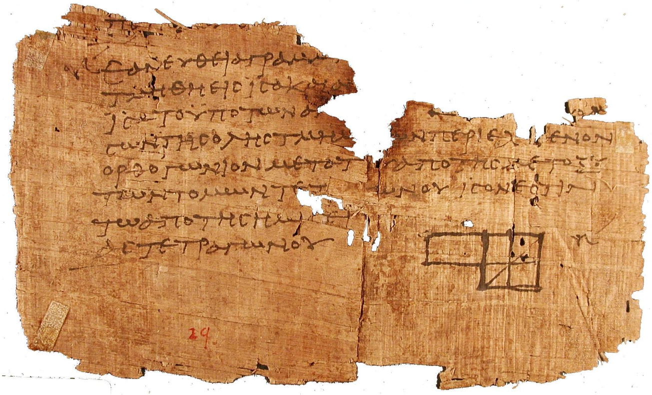
Geometry: from past to future...



Inaugural lecture for the 2021 Francqui Chair at the ULB

Introduction to geometry

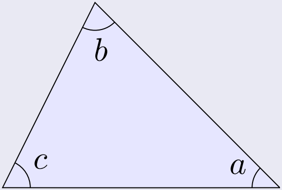
Euclidean geometry was created by Euclid in 300 BC and written down in a series of 13 books called “The Elements”.



Euclidean geometry

- Euclid deals with familiar concepts such as points, lines, lengths and angles and proves that these have the expected properties.

Theorem (The Elements, Book I, Proposition 32)



$$a + b + c = 180^\circ$$

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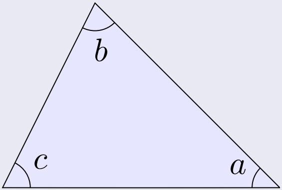

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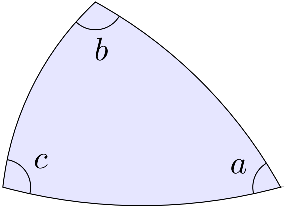
- The Elements (300BC) were written in modern mathematical style with Axioms, Definitions, Theorems and Proofs. The axioms, from which everything else starts, were regarded as self-evident truths.

Euclidean geometry and reality

- It would be nice if reality were indeed described by Euclidean geometry, but things are not that simple. For example the Earth's surface is a sphere (Aristotle 330BC) so it is not described by Euclidean geometry.



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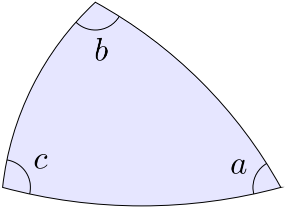
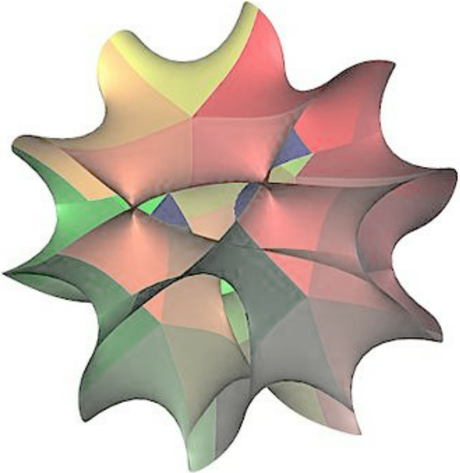


$$a + b + c > 180°$$

- In fact the universe as a whole is described by a 4-dimensional curved space (Albert Einstein 1915).

Euclidean geometry and reality

String theory even uses spaces with many more dimensions with a very complicated geometry.



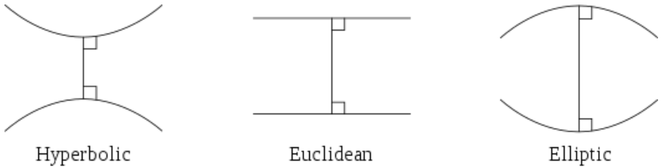
Two dimensional cross section of a quintic 3-fold

A non-Euclidean space in art

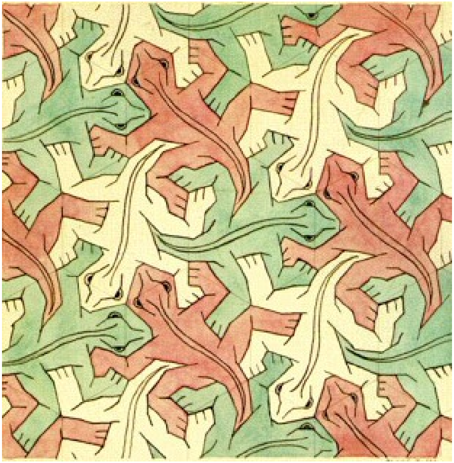
Circle Limit III (M.C. Escher 1959)

The hyperbolic plane

The hyperbolic plane is a kind of geometry where a different version of the infamous 5'th postulate by Euclid holds.



Euclidean analogue of Circle Limit III



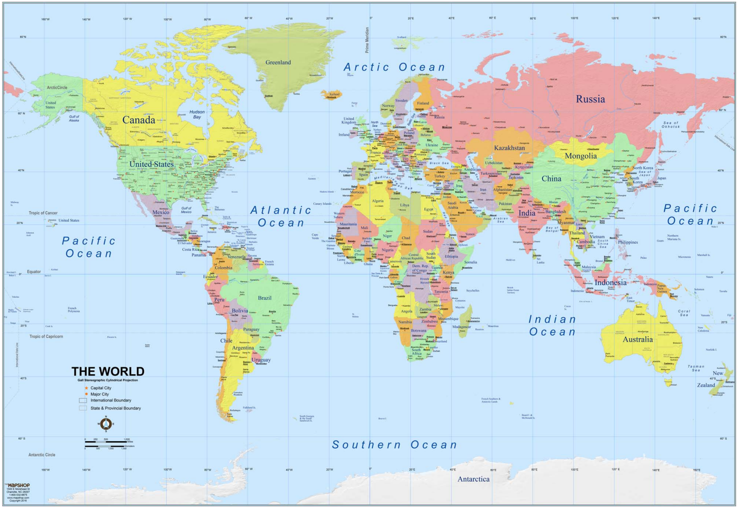
Study of Regular Division of the Plane with Reptiles (M.C. Escher 1939)

The hyperbolic plane in art

Circle Limit III (M.C. Escher 1959)

Charts and atlases

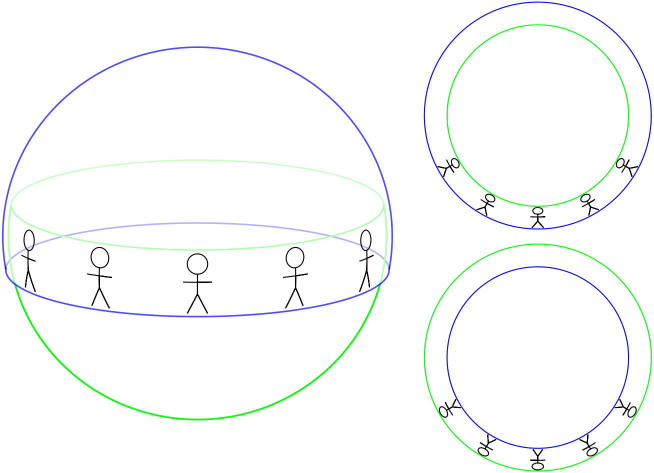
Like real world carthographers, mathematicians use the concepts of "charts" and "atlases" to describe the spaces they use.



Charts and atlases

Charts and atlases

An atlas of a space is a collection of overlapping charts which cover the whole space.



From charts and atlases to geometry

- Depending on the types of charts allowed, and the ways of gluing them together ("coordinate changes"), many different types of spaces can be created. These are often called manifolds.

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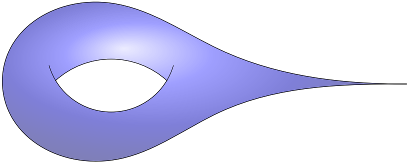
- Physics uses the language of symplectic geometry in which for example a 1-dimensional particle is described by its position coordinate (q) and momentum coordinate $(p = \text{mass} \times \text{speed})$. Coordinate changes should respect the "symplectic form" $dq \wedge dp$.

Complex versus real versus symplectic geometry: example

- For $\lambda \in \mathbb{C} - \{0, 1\}$ consider

  $$E_\lambda := \{(x, y) \in \mathbb{C}^2 \mid y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{C}^2 (\cong \mathbb{R}^4)$$

(real structure independent of λ)



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  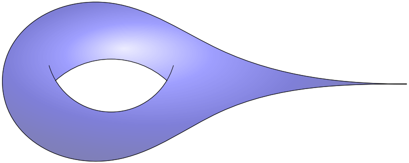

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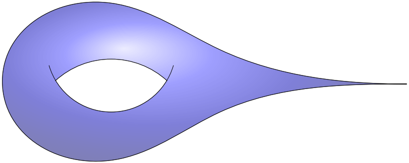

- The E_λ inherit a complex structure and a symplectic structure from $\mathbb{C}^2$.
- The E_λ are different as complex manifolds (for generic values of λ).
- On the other hand the E_λ are all identical as symplectic manifolds.

The birth of mirror symmetry

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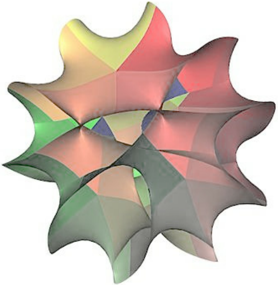

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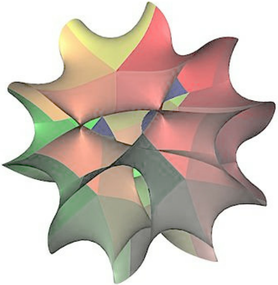

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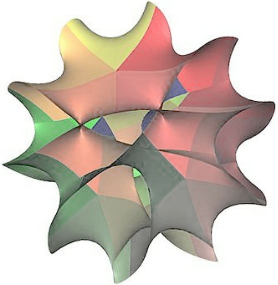

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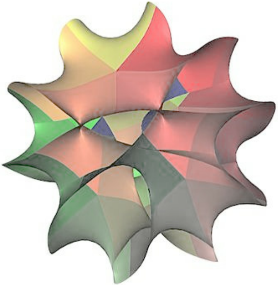

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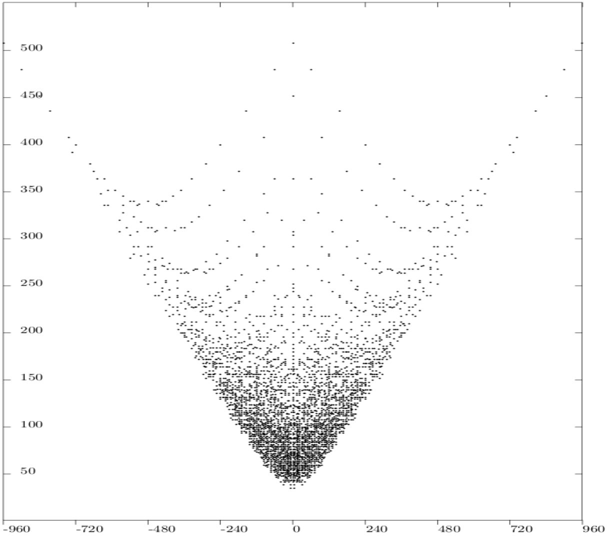
- Mirror symmetry was born.

Mirror symmetry



Philip Candelas, Xenia de la Ossa and Sheldon Katz (1994)

Applications of mirror symmetry

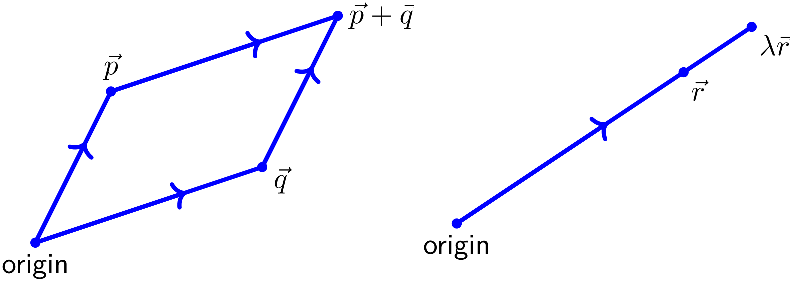
- Originally enumerative applications. E.g. rational curves on the quintic threefold (P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, 1991 - Morrison, 1992).
- "Homological mirror symmetry" (Maxim Kontsevich, 1994).

The plan

- Introduction (done)
- Vector bundles
- Examples of vector bundles
- Mirror symmetry for vector bundles on the two dimensional sphere
- The Strominger-Yau-Zaslow conjecture
- More examples

Vector spaces

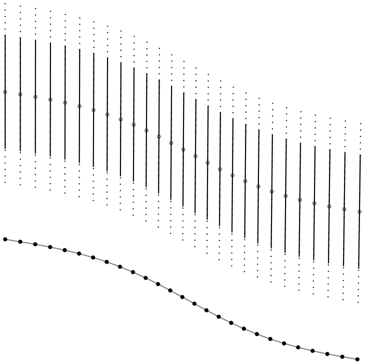
A vector space is a space equipped with addition and scaling.



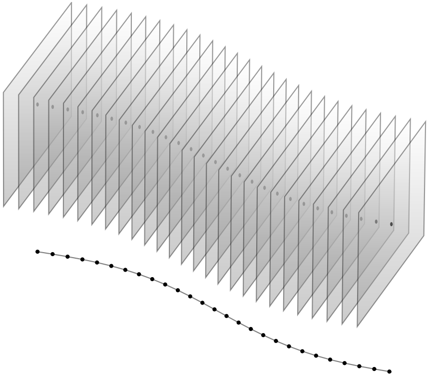
Examples: a line, or a plane (with a marked origin)

Vector bundles

Informally a vector bundle is a collection of vector spaces parametrized by a space.



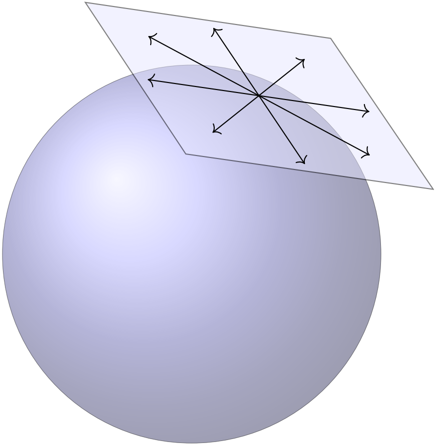
A vector bundle of rank 1
(a line bundle)



A vector bundle of rank 2

Example of a vector bundle: the "tangent bundle"

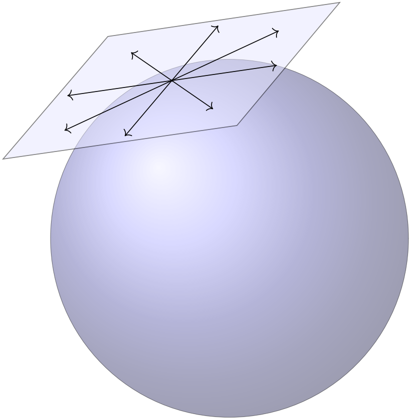
A tangent space is a local approximation of a space at a point by a vector space.



An element of the tangent space is called a tangent vector.

Example of a vector bundle: the "tangent bundle"

The tangent space moves if the point moves.



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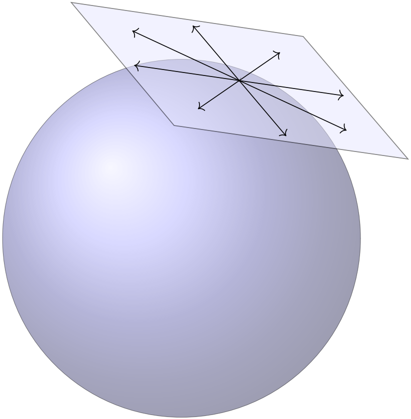

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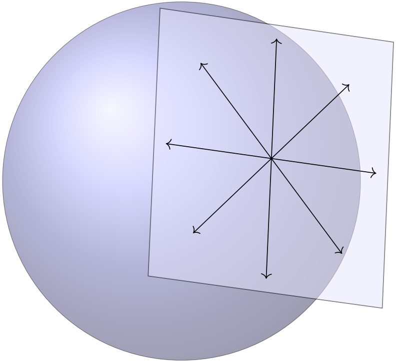

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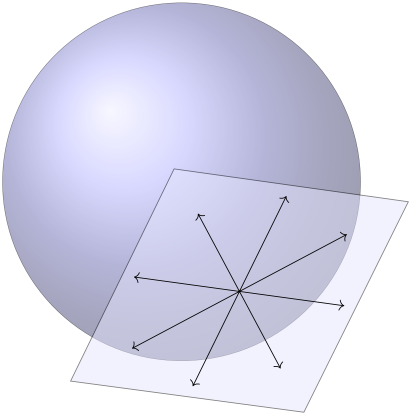

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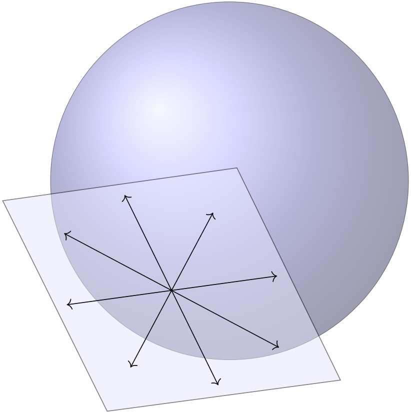

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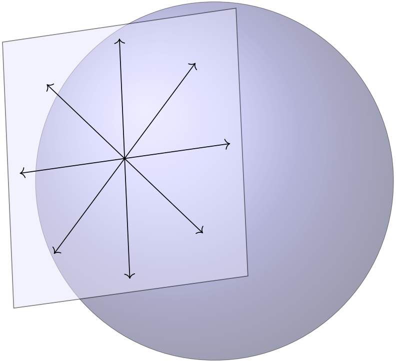

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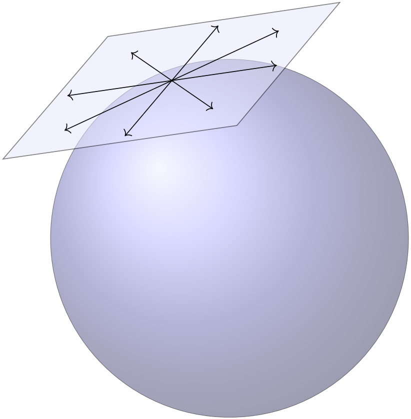

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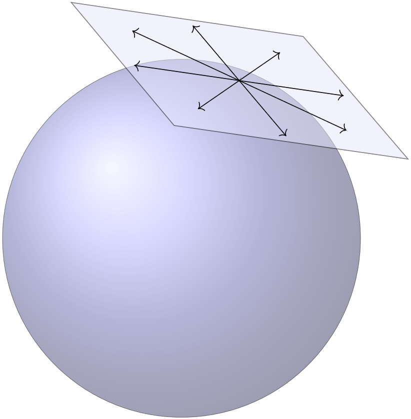

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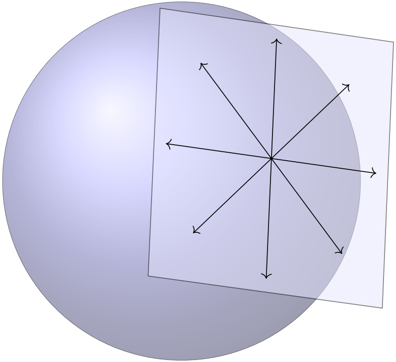

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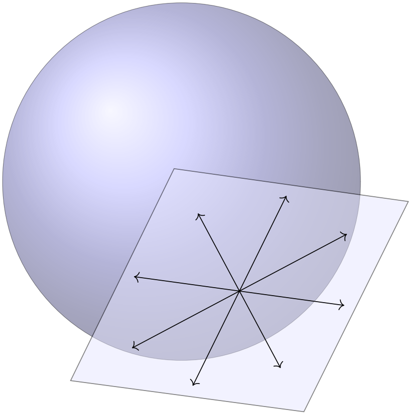

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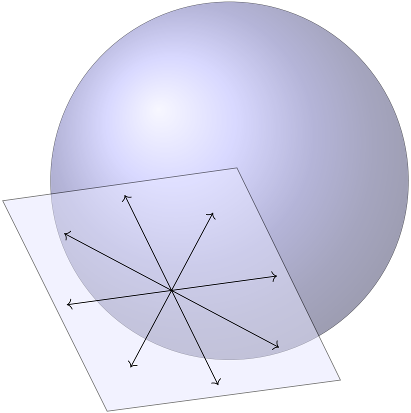

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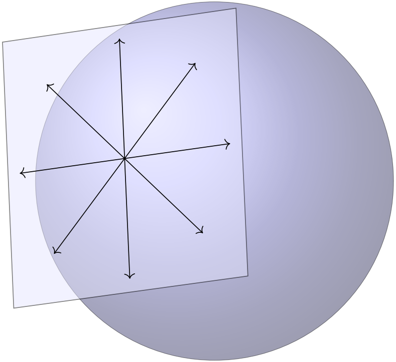

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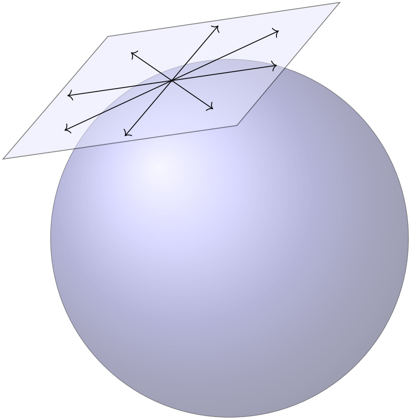
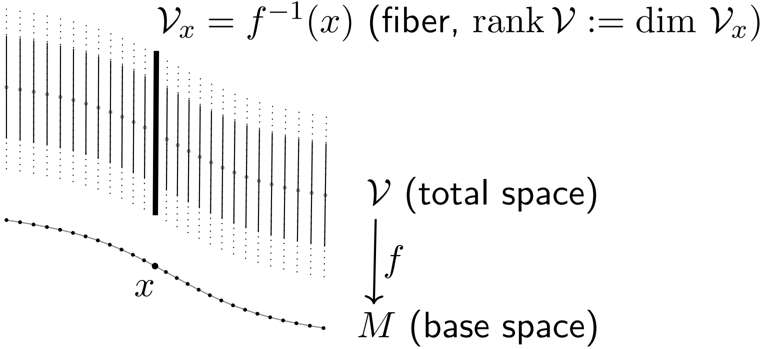


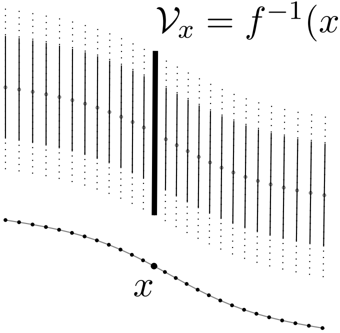
The disjoint union of all the tangent spaces is called the tangent bundle. It is an example of a vector bundle.

Vector bundles



Vector bundles

$\mathcal{V}_x = f^{-1}(x)$ (fiber, $\operatorname{rank} \mathcal{V} := \dim\, \mathcal{V}_x$)



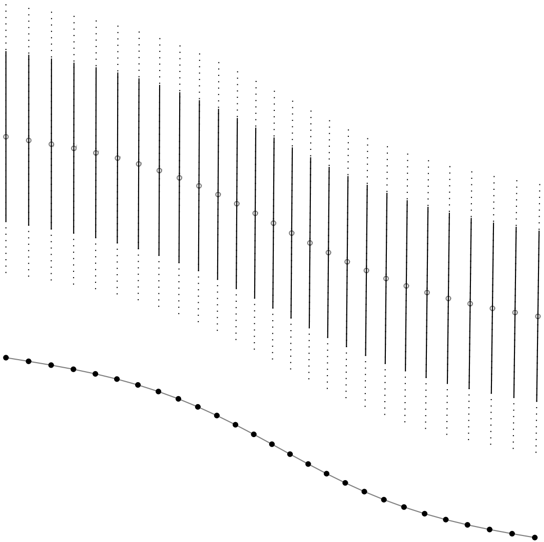
$$\mathcal{V} \text{ (total space)} \xrightarrow{f} M \text{ (base space)}$$

A vector bundle is said to be (globally) trivial if it is of the form

$$\mathcal{V} = \mathbb{R}^r \times M \xrightarrow{\text{projection}} M$$

Do non-trivial vector bundles exist? If so, how to recognize them?

Sections



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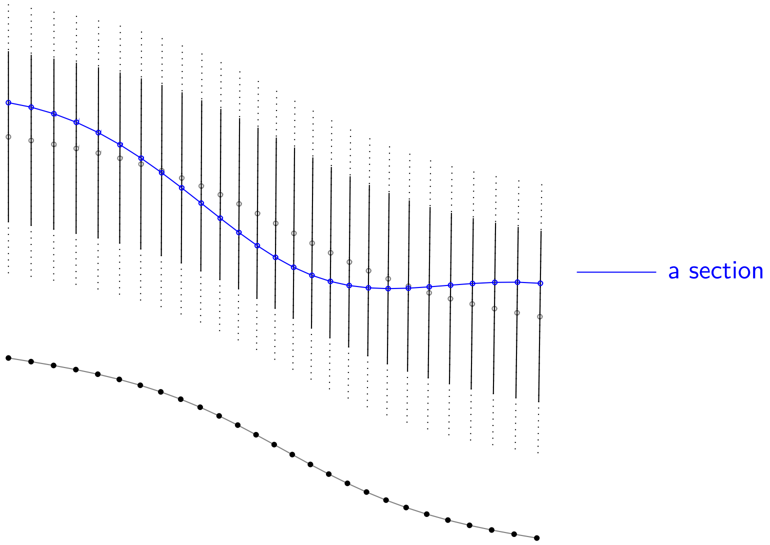

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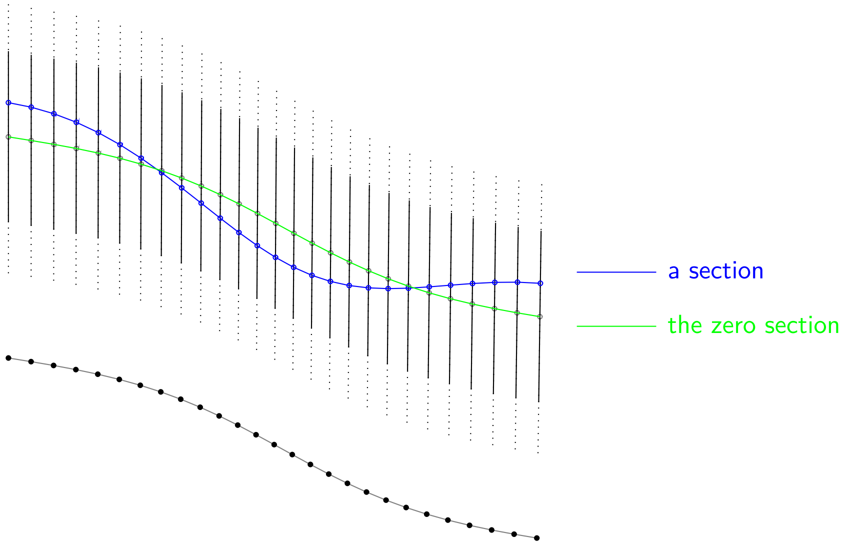

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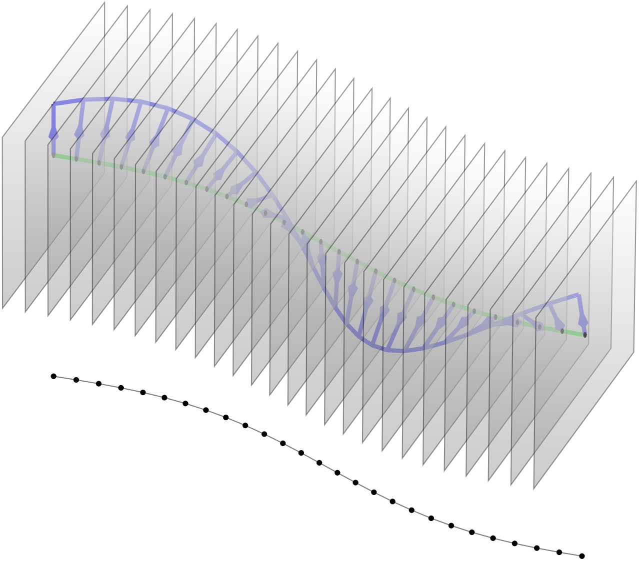

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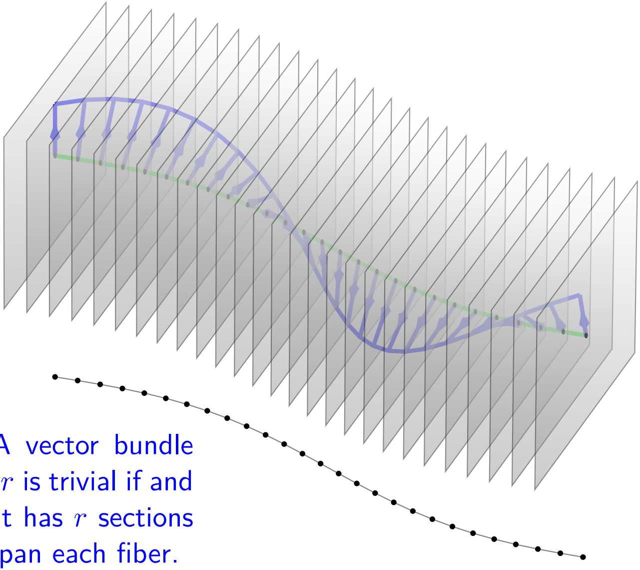
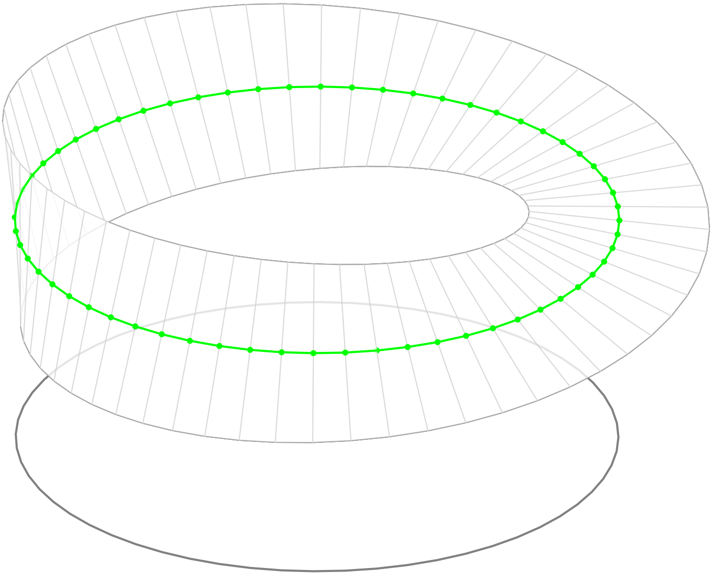


Fact: A vector bundle of rank r is trivial if and only if it has r sections which span each fiber.

The Möbius strip: a non-trivial line bundle on the circle



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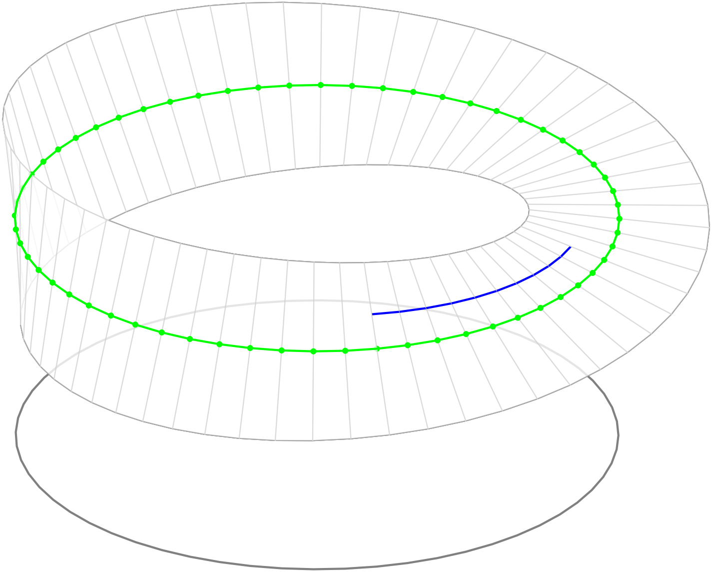

Trying to construct a non-zero section...

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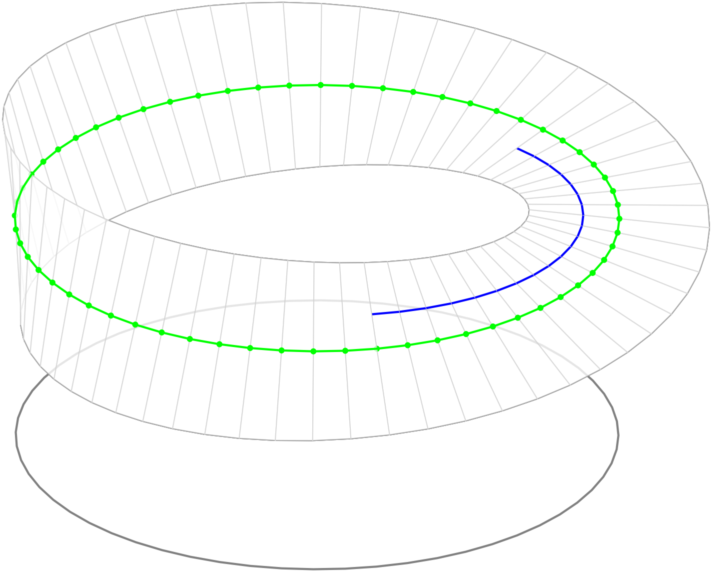

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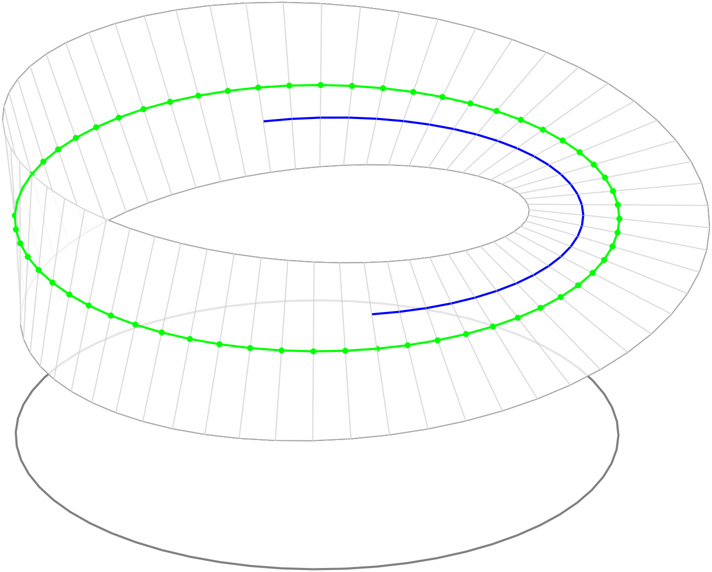

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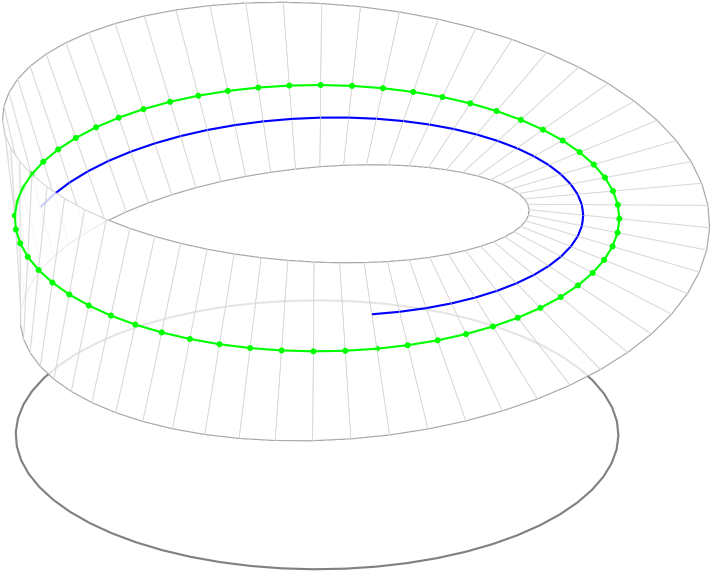

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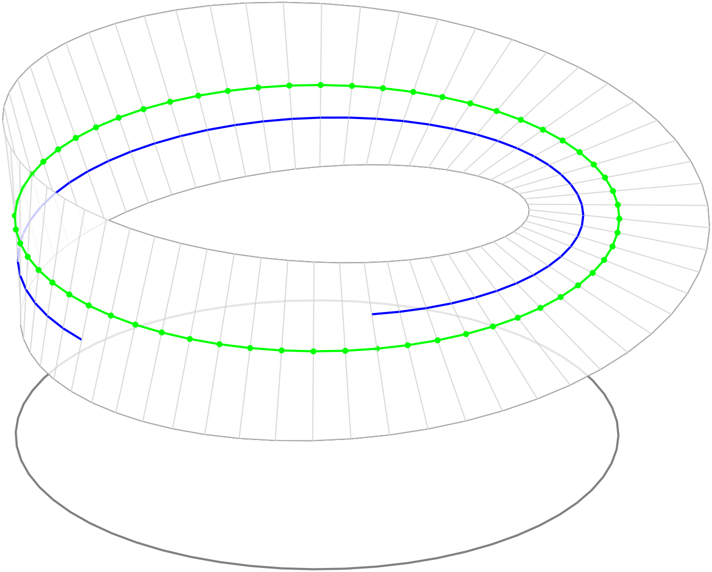

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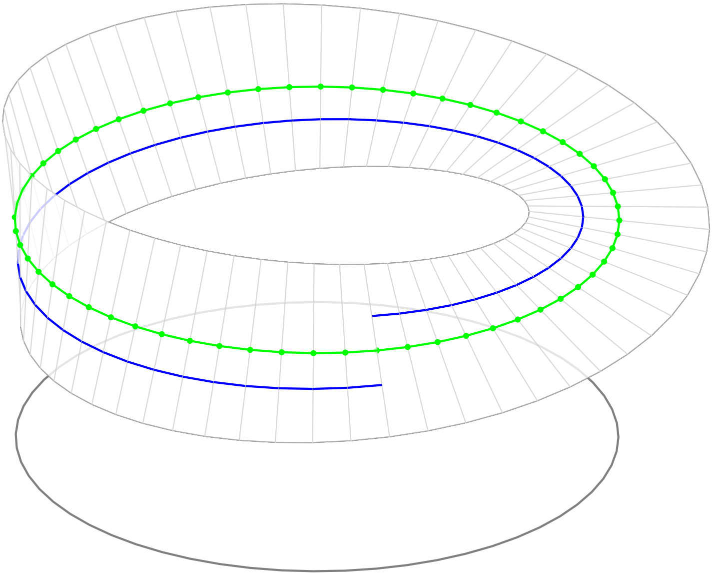

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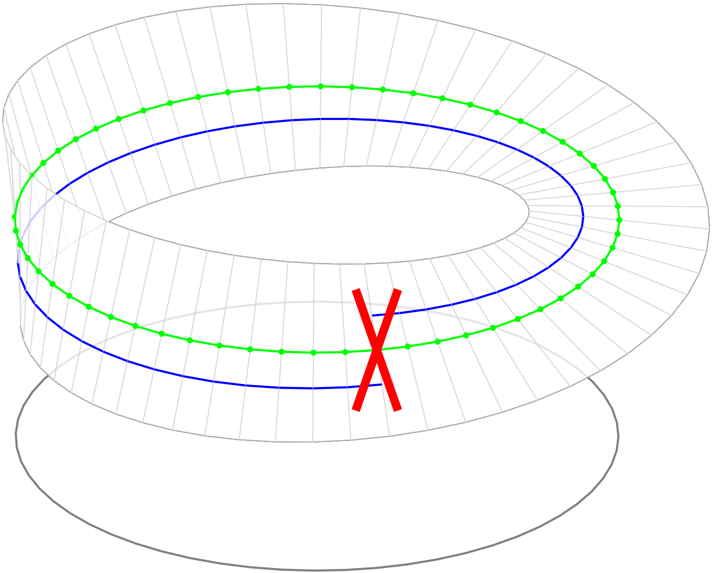


Oops...

The tangent bundle is often non-trivial

- The sections of the tangent bundle are called vector fields.
- In other words: A vector field is a choice of tangent vector in every point which varies "smoothly" with the point.

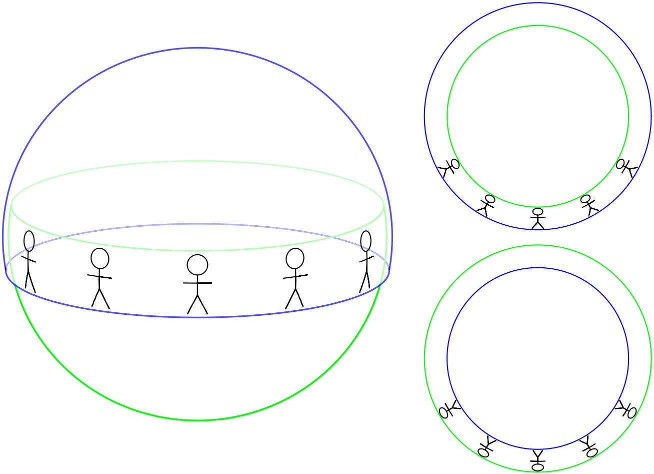
The hairy ball theorem

- The hairy ball theorem states that there is no vector field on a sphere that is (everywhere) non-zero.
- Informally: "You cannot comb the hair on a coconut".

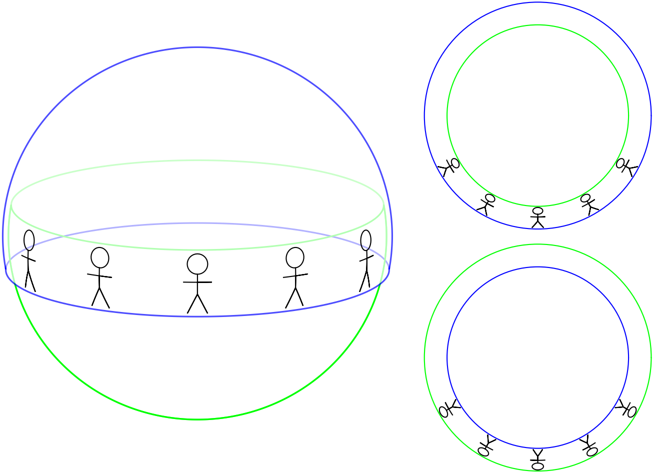
- In other words: the tangent bundle of a sphere is non-trivial.

Proof of the hairy ball theorem

Choose the "convenient atlas".

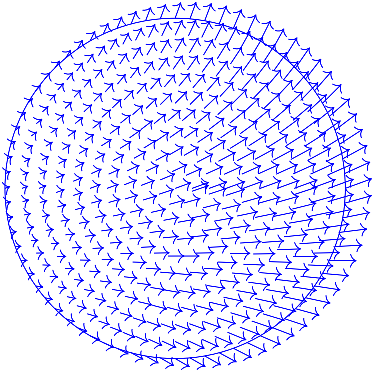


Proof of the hairy ball theorem (by contradiction)



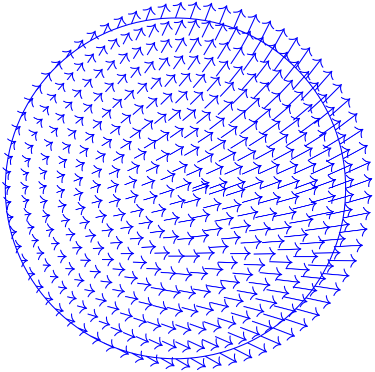
A non-zero vector field on the sphere may be described by non-zero vector fields on the northern and southern hemisphere (disks) which agree on the equator.

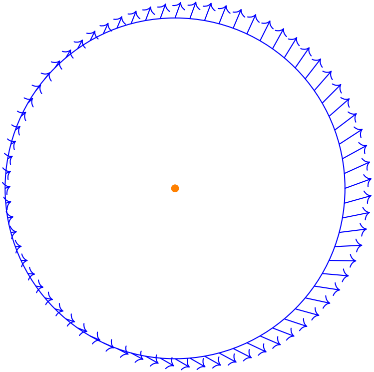
Non-zero vector fields on the northern hemisphere



(an example of a non-zero northern vector field)

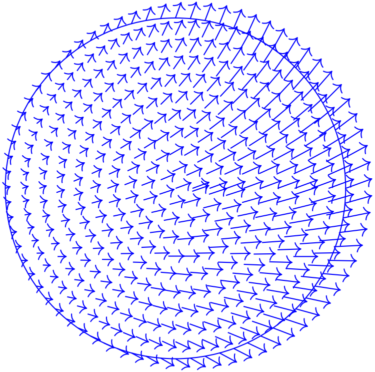
Non-zero vector fields on the northern hemisphere

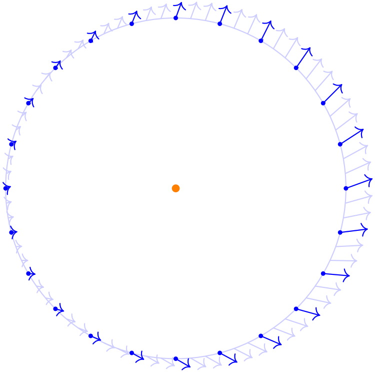




(the equator vector field)

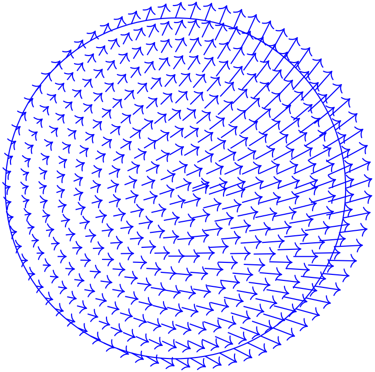
Non-zero vector fields on the northern hemisphere

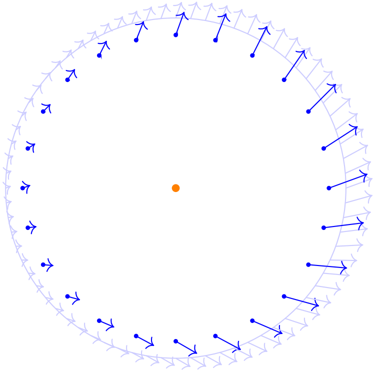




(translating/scaling the equator vectors to the center…)

Non-zero vector fields on the northern hemisphere





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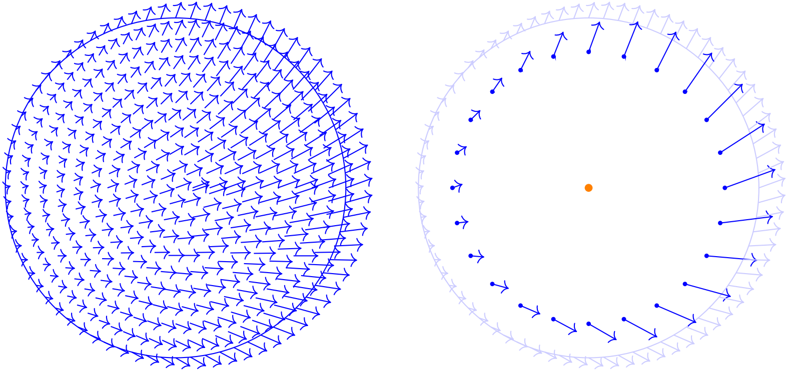

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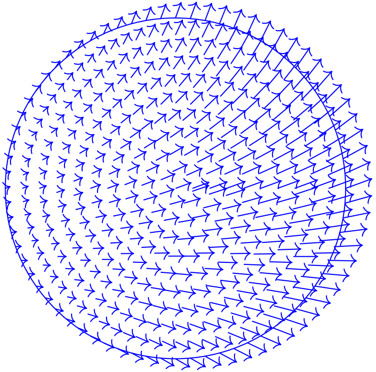 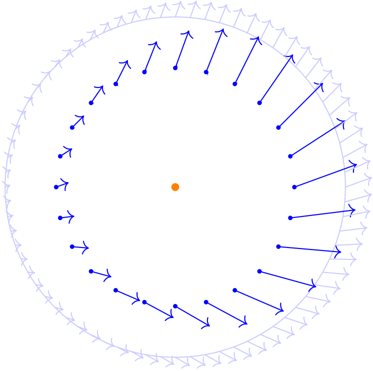

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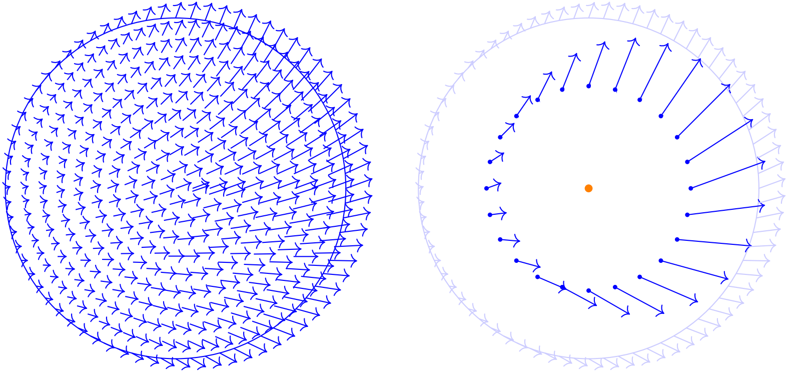

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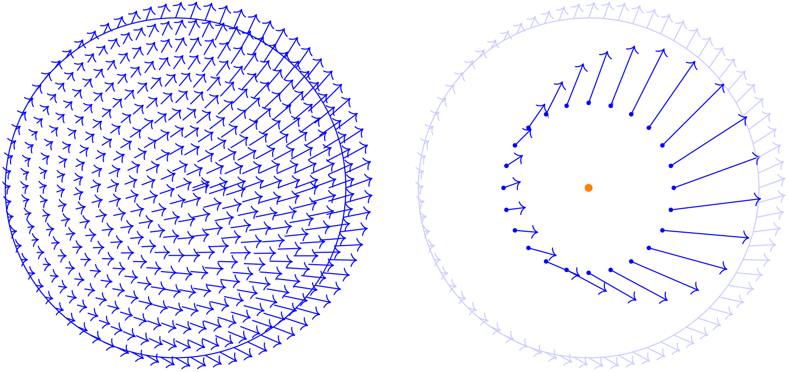

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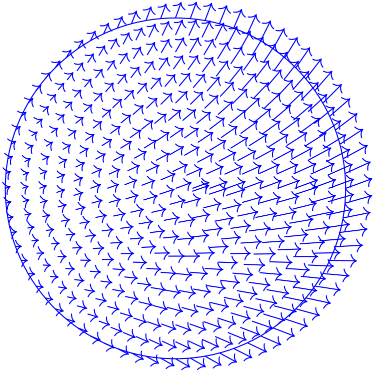

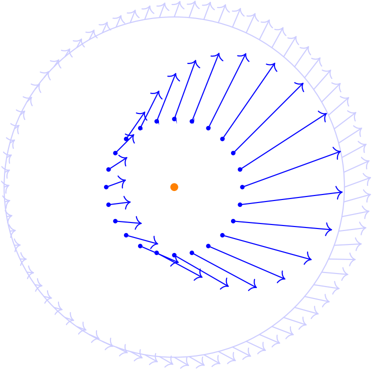

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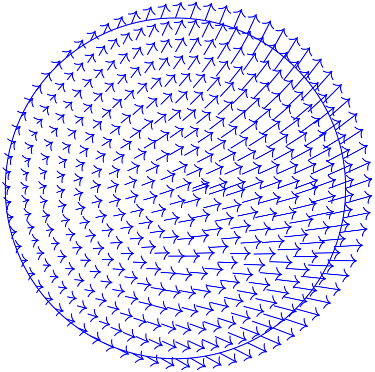

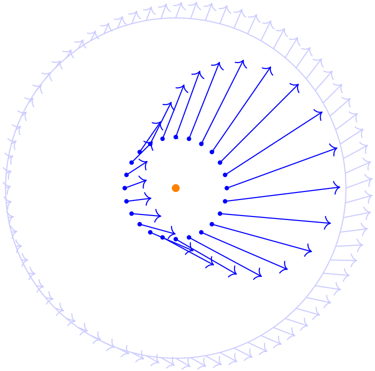

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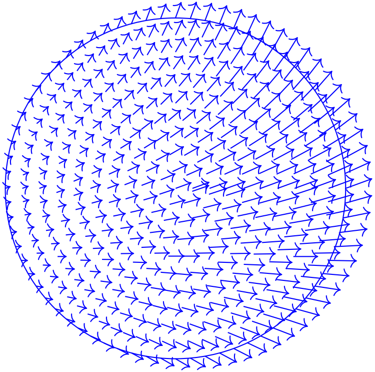

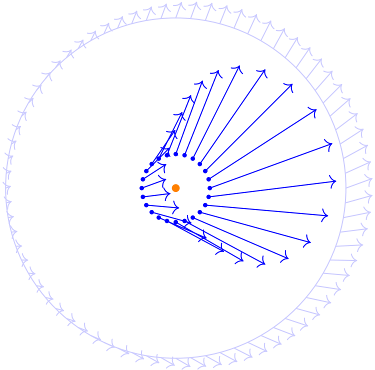

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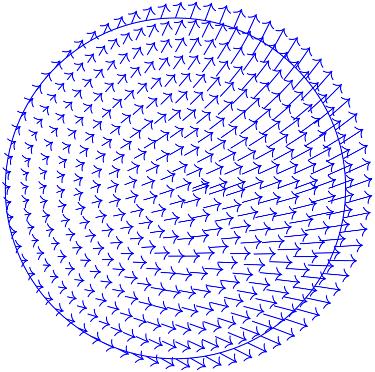

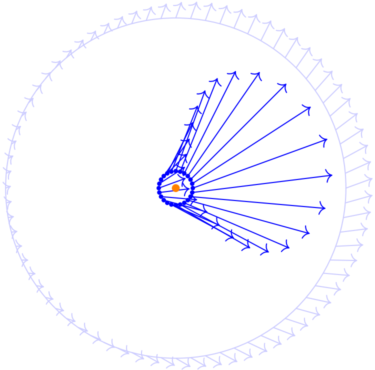

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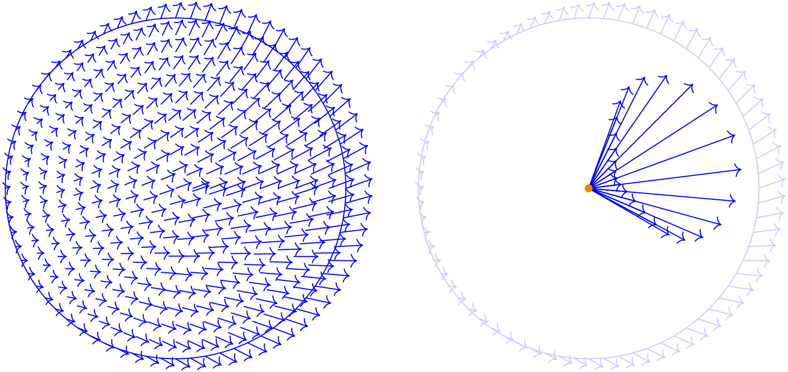

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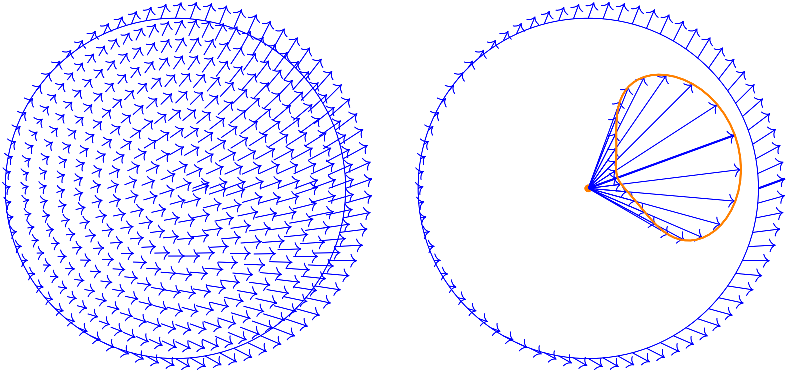
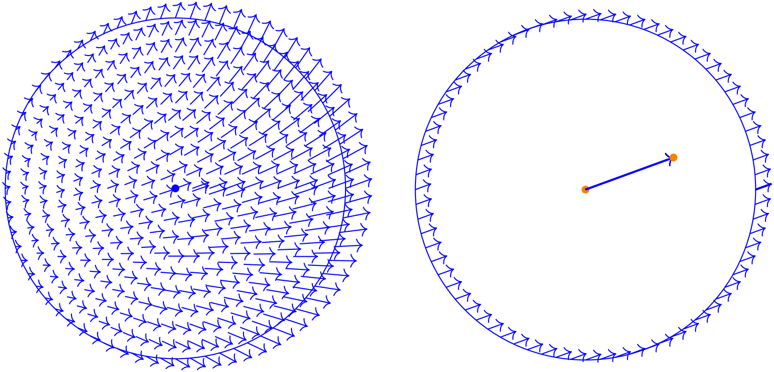


(finished...)

- The vectors on the boundary twist and turn but they don't wind around the origin (cfr the orange curve). This can be seen by shrinking the boundary to a point.

Non-zero vector fields on the northern hemisphere



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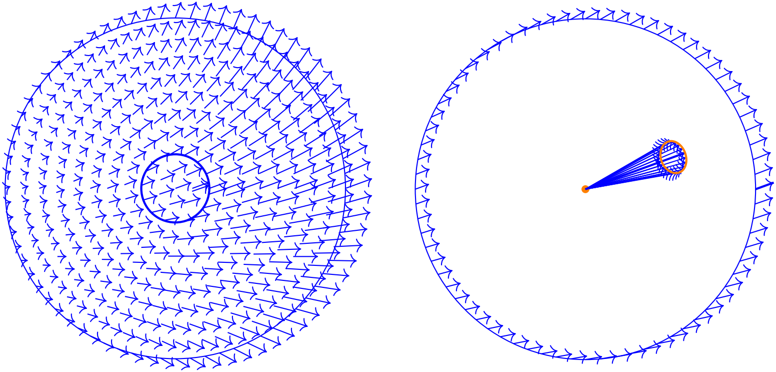

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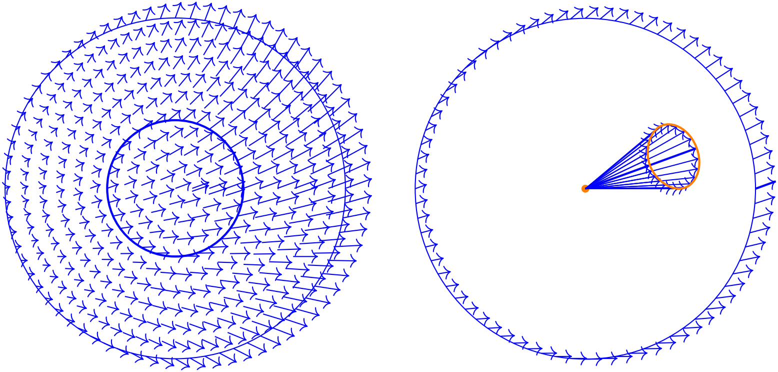

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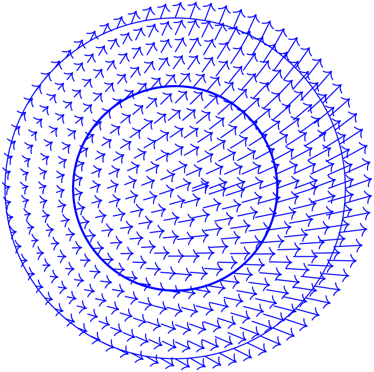

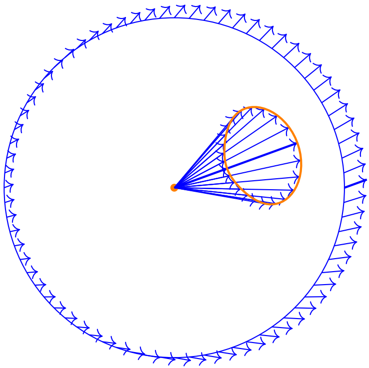

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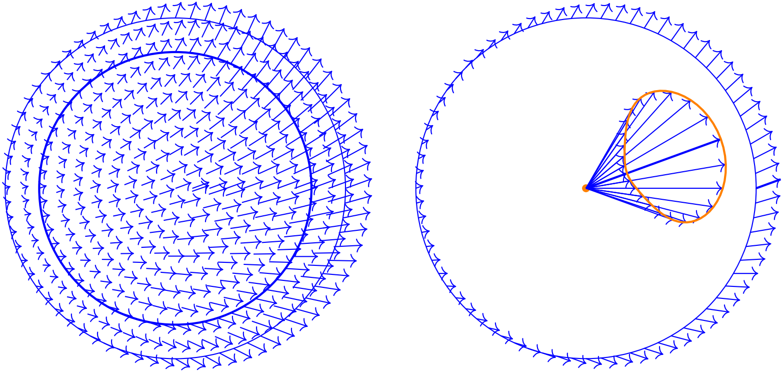
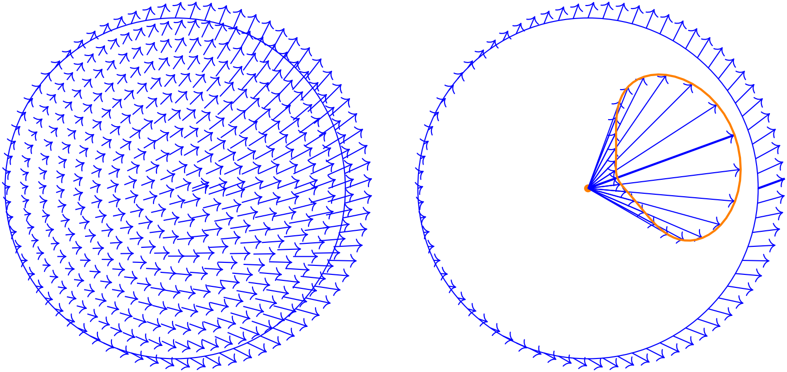


(shrunken boundary)

- The vectors on the boundary twist and turn but they don't wind around the origin (cfr the orange curve). This can be seen by shrinking the boundary to a point. The orange curve can never cross the origin because the vector field on the disk is everywhere non-zero.

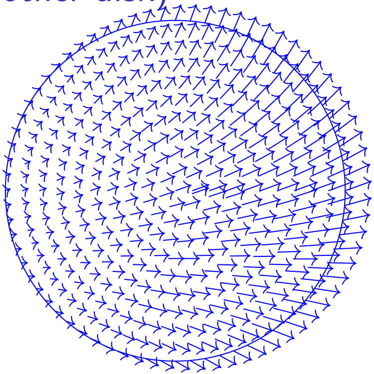
Non-zero vector fields on the northern hemisphere



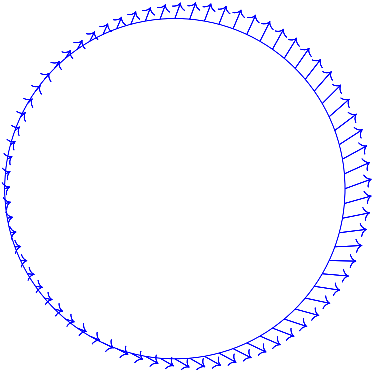
(actual boundary)

- The vectors on the boundary twist and turn but they don't wind around the origin (cfr the orange curve). This can be seen by shrinking the boundary to a point. The orange curve can never cross the origin because the vector field on the disk is everywhere non-zero.

Matching the northern and the southern hemisphere (another disk)



Northern hemisphere



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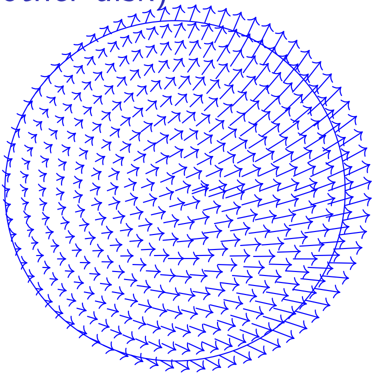
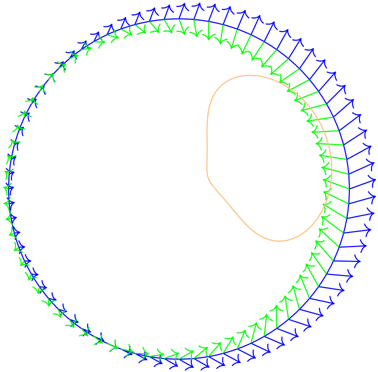
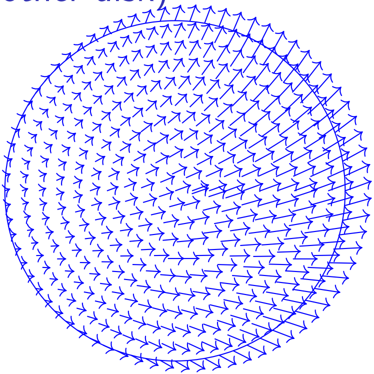


Northern hemisphere

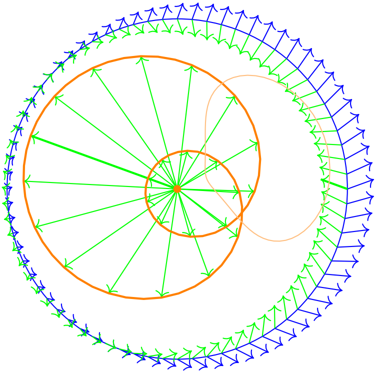


Southern hemisphere

Matching the northern and the southern hemisphere (another disk)



Northern hemisphere

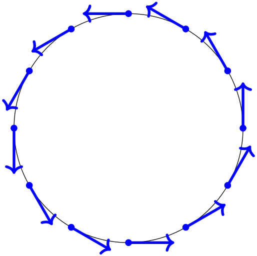


Southern hemisphere

- The tangent vectors on the boundary of the southern hemisphere wind around the origin (the dark orange curve) because they incorporate the rotation of the circle.
- Hence they cannot be obtained from a non-zero vector field on the southern hemisphere. The hairy ball theorem is proved.

Other spaces

- Observation: the 1-dimensional sphere (circle) has a non-zero vector field.



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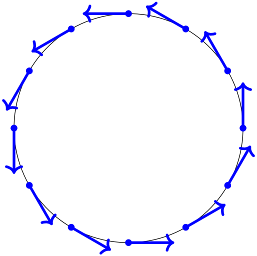

- Facts: a sphere admits a non-zero vector field if and only if it has odd dimension.

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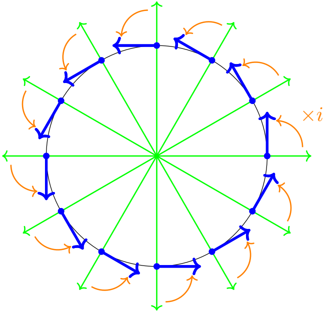
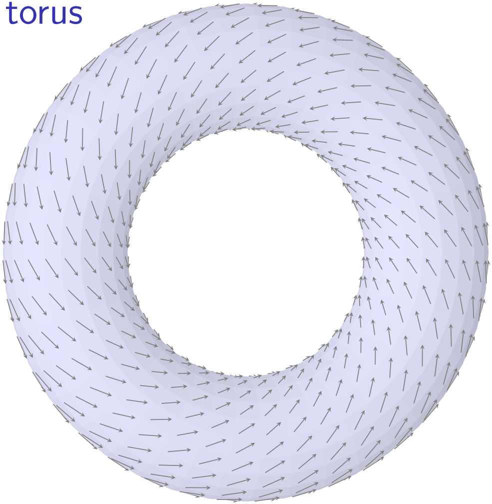


- Facts: a sphere admits a non-zero vector field if and only if it has odd dimension.
- The only spheres with trivial tangent bundle are those of dimension 1, 3 and 7. This is intimately connected with the existence of the quaternions and the octonions.

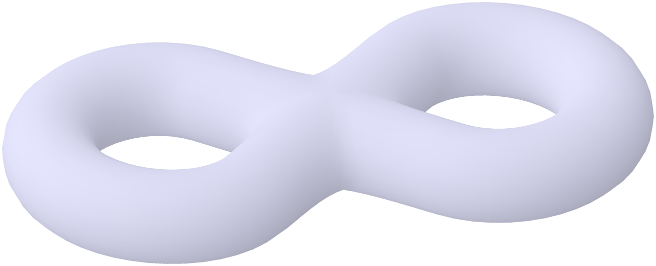
Example: the torus



- A torus is the same as circle $\times$ circle.
- The tangent bundle on the circle is trivial, hence so is the tangent bundle on the torus.

Surfaces with more holes

- (Compact, orientable) surfaces are characterized by their number of holes. This is called "the genus" (below an example of a genus 2 surface).



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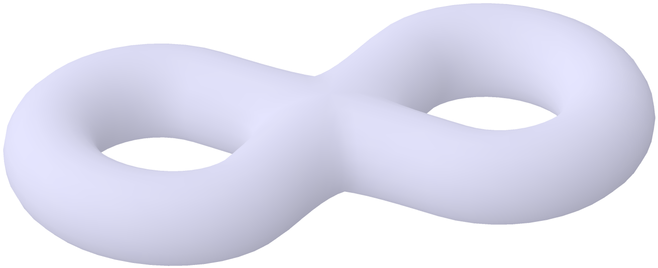

- Surfaces of genus $\neq 1$ do not admit non-zero vector fields. Hence their tangent bundles are non-trivial.

The direct sum of vector spaces and vector bundles

- Vector spaces have a natural "direct" sum operation.

$$V \oplus W := V \times W = \{(v, w) \mid v \in V, w \in W\}.$$

One has

$$\dim(V \oplus W) = \dim V + \dim W.$$

The direct sum of vector spaces and vector bundles

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One has

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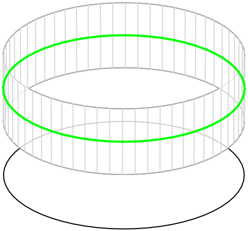
- For vector bundles $\mathcal{V}$, $\mathcal{W}$ there is a similar direct sum $\mathcal{V} \oplus \mathcal{W}$, defined fiberwise via

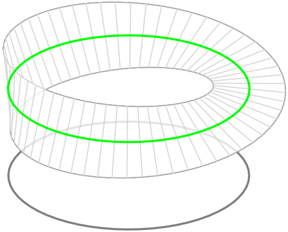
$$(\mathcal{V} \oplus \mathcal{W})_x = \mathcal{V}_x \oplus \mathcal{W}_x.$$

Vector bundles on the circle

- We know of two line bundles on the circle:



$\mathcal{O}$ (the trivial bundle $\mathbb{R} \times S^1$)



$\mathcal{M}$ (the Möbius strip)

Vector bundles on the circle

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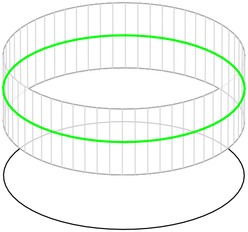

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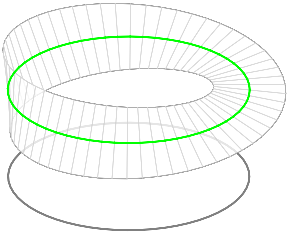

$\mathcal{M}$ (the Möbius strip)

- Fact: Every vector bundle of rank k on the circle is of the form

$$\underbrace{\mathcal{O}\oplus\cdots\oplus\mathcal{O}}_{k} \qquad \text{or} \qquad \underbrace{\mathcal{O}\oplus\cdots\oplus\mathcal{O}}_{k-1}\oplus\mathcal{M}.$$

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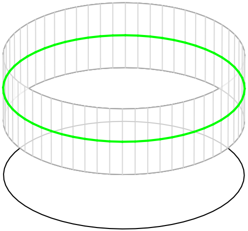

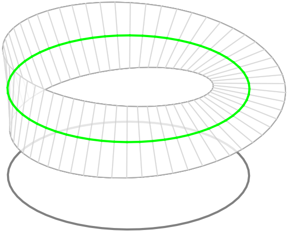

$\mathcal{O}$ (the trivial bundle $\mathbb{R}\times S^1$) $\qquad$ $\mathcal{M}$ (the Möbius strip)

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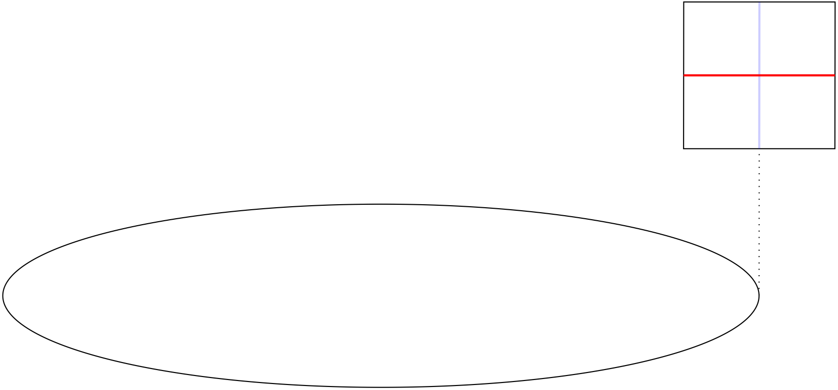
- Fact: There is an "interesting relation"

$$\mathcal{M} \oplus \mathcal{M} \cong \mathcal{O} \oplus \mathcal{O}.$$

Observation

The Möbius strip $\mathcal{M}$ can be embedded in the trivial bundle $S^1 \times (\mathbb{R} \oplus \mathbb{R}) = \mathcal{O} \oplus \mathcal{O}$ of rank 2.



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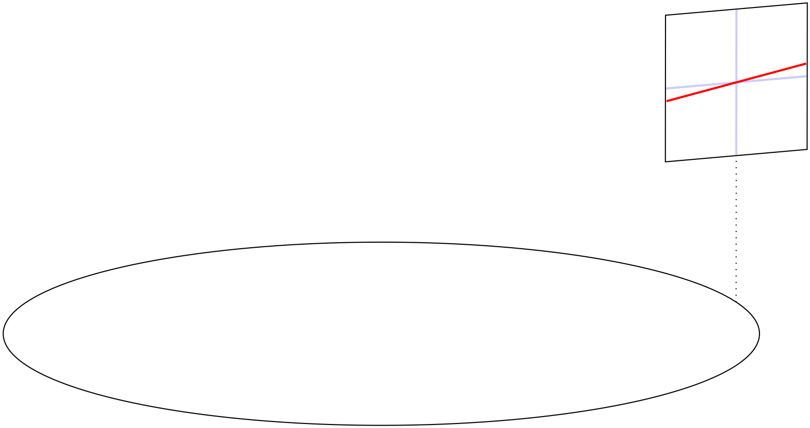

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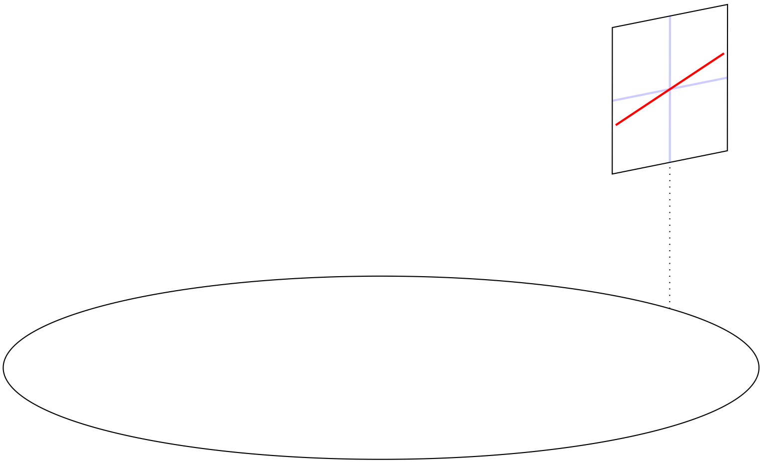

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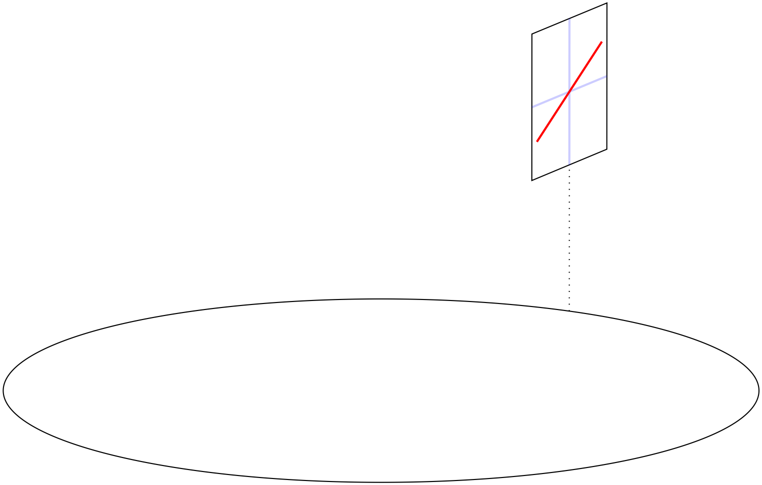

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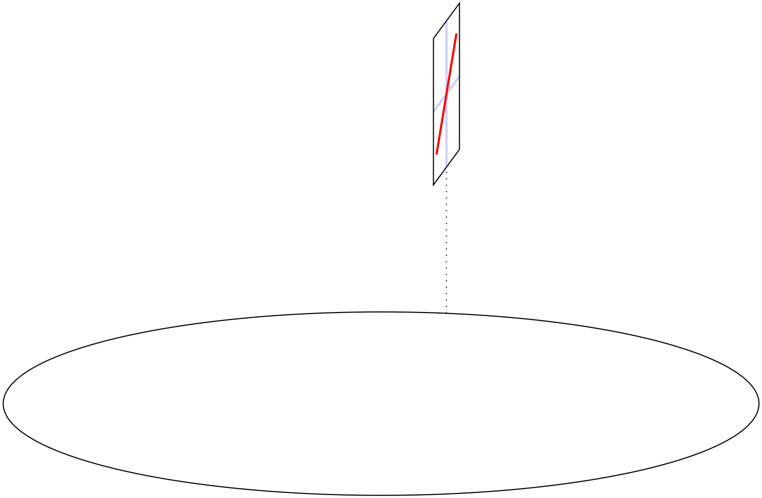

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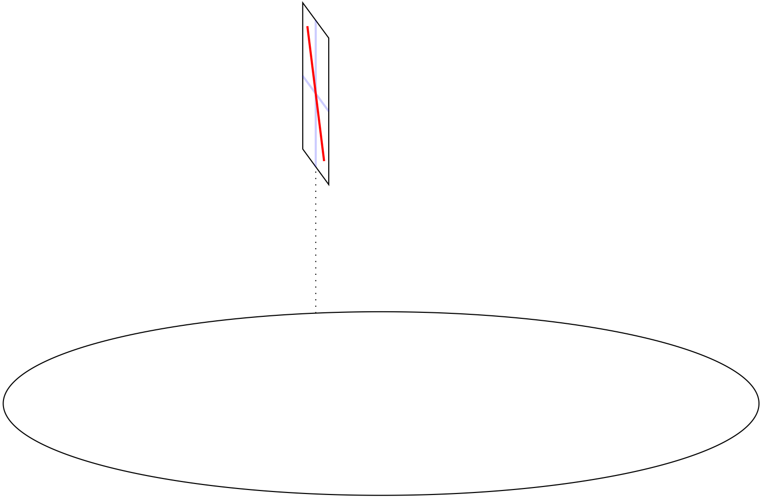

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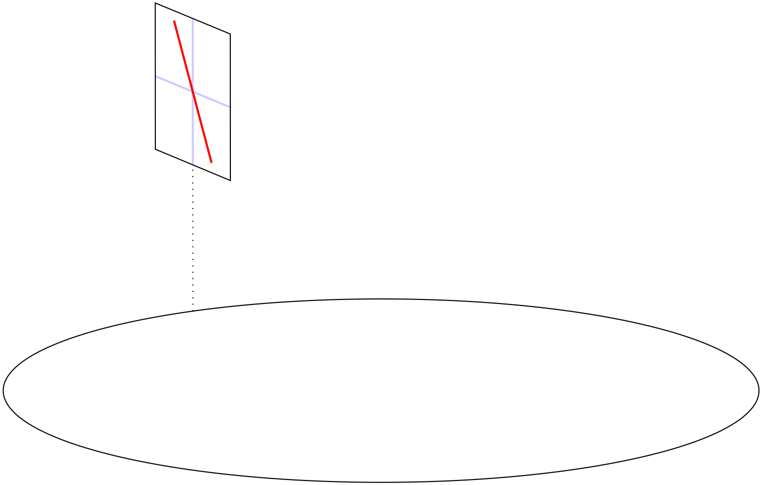

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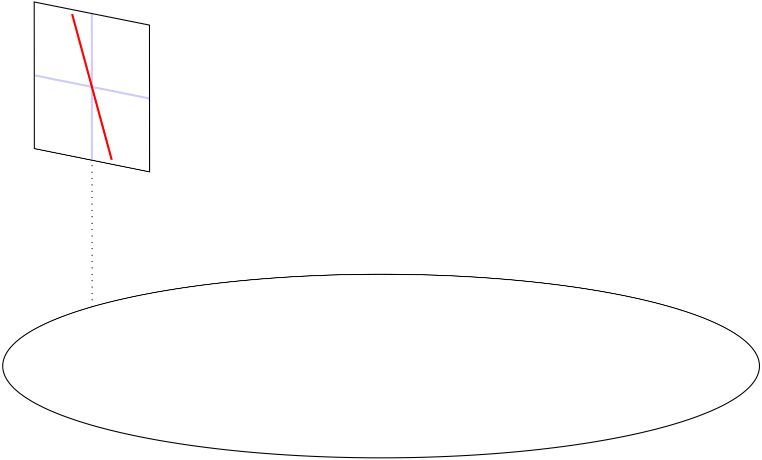

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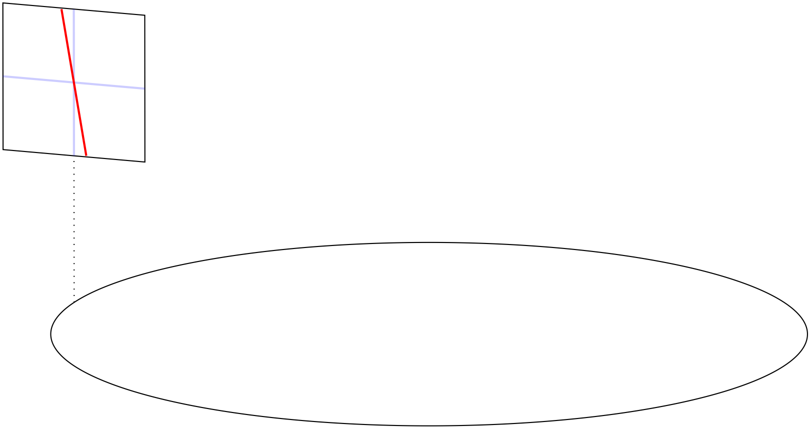

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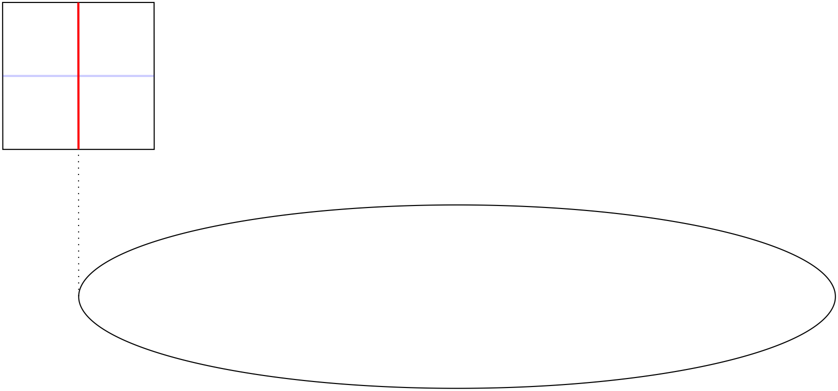

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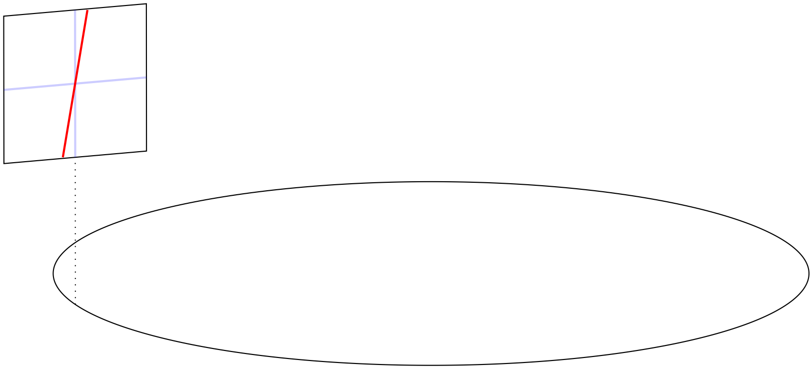

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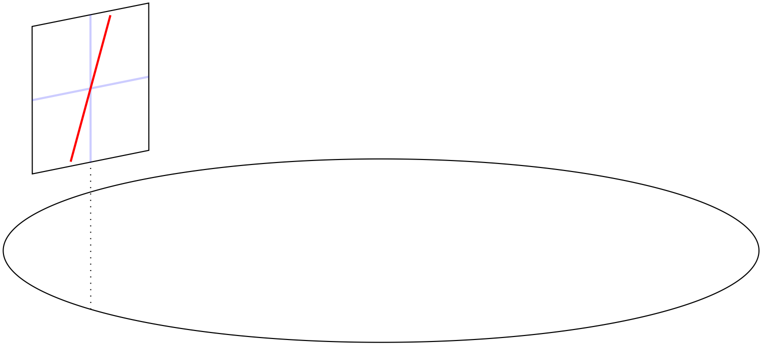

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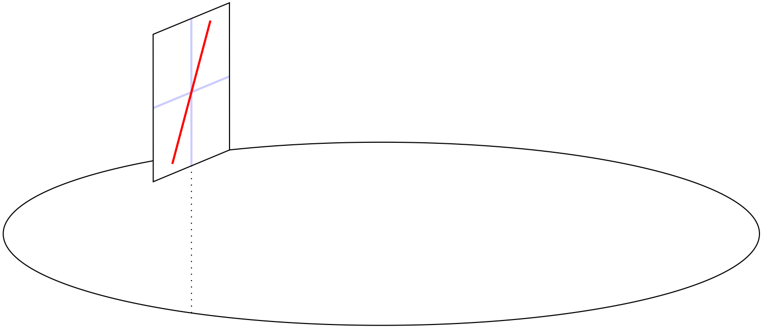

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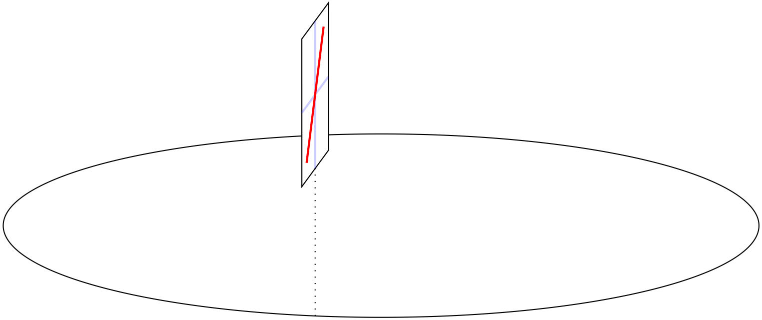

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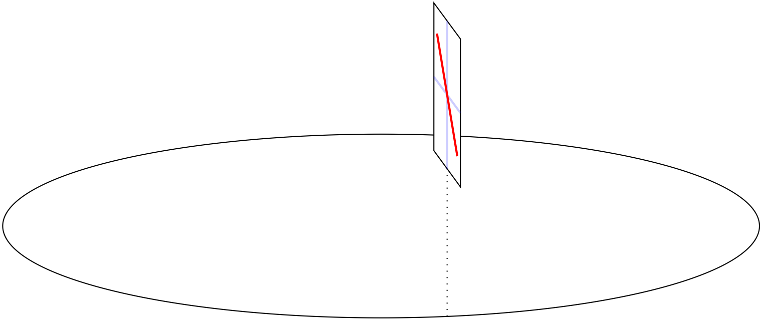

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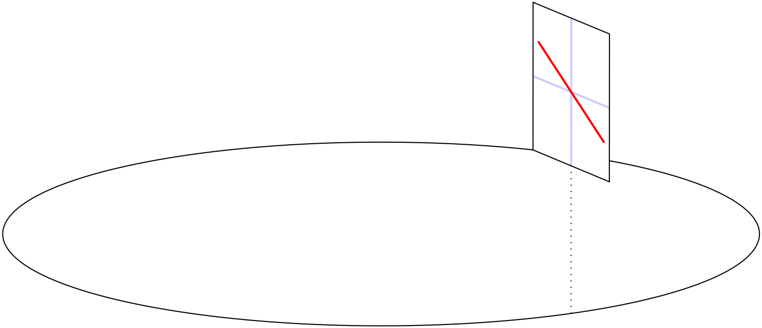

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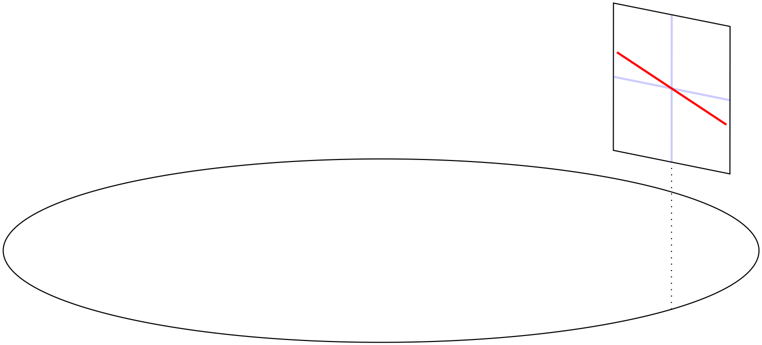

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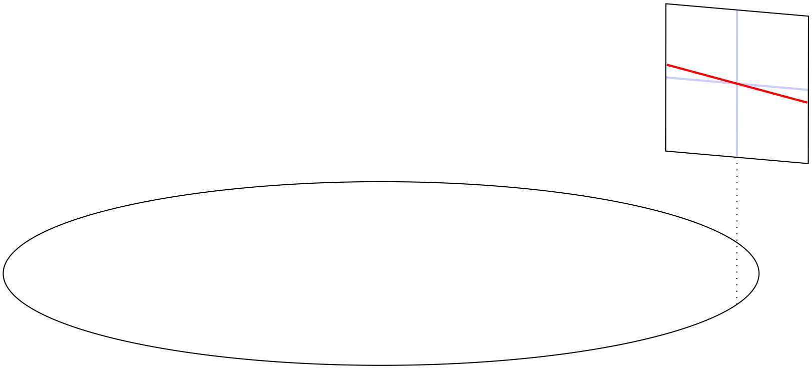

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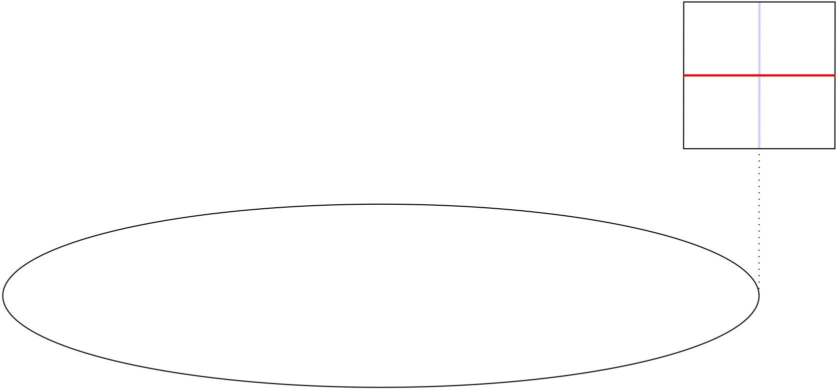

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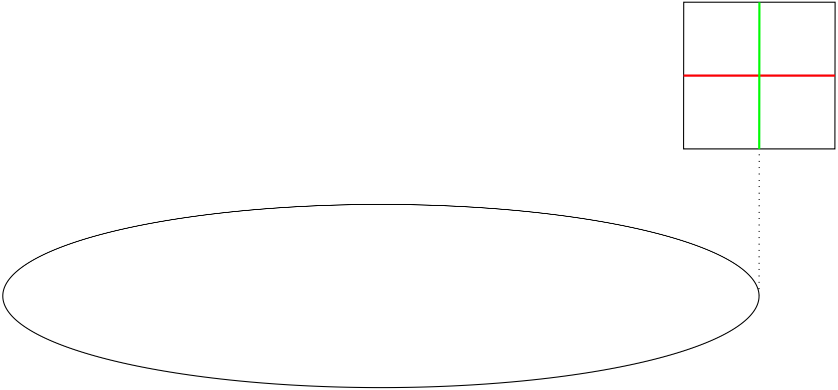

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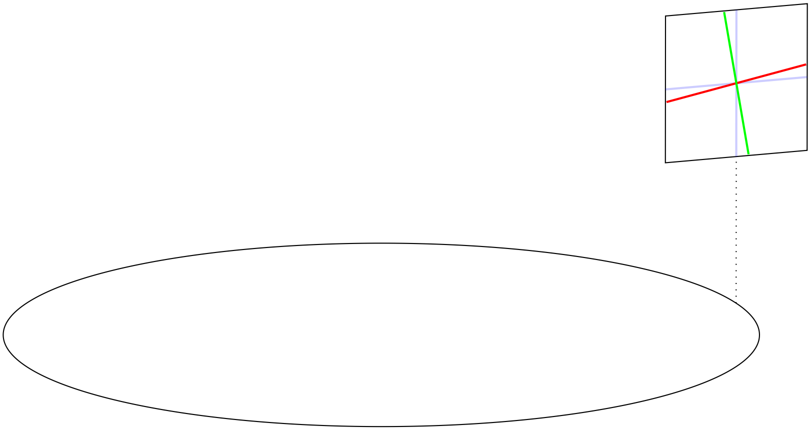

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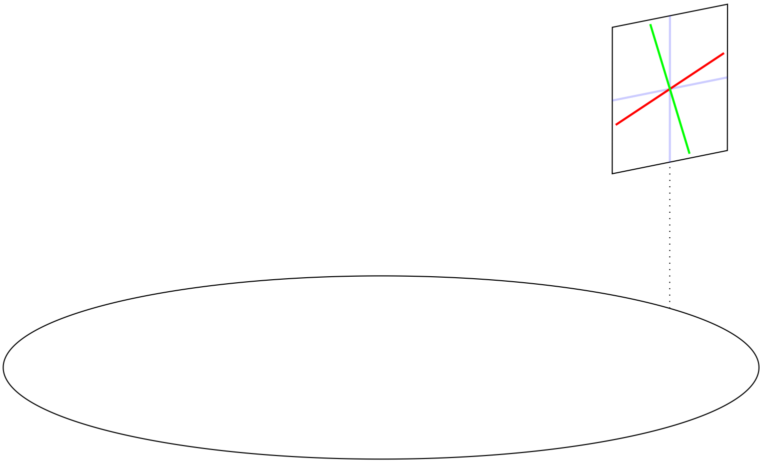

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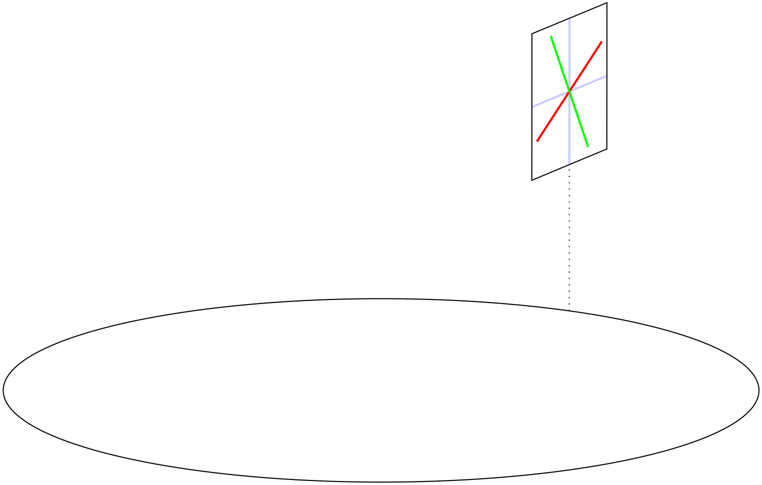

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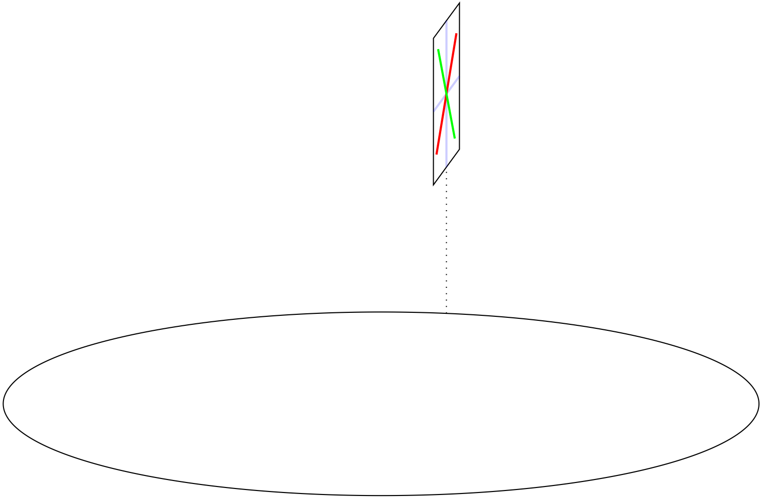

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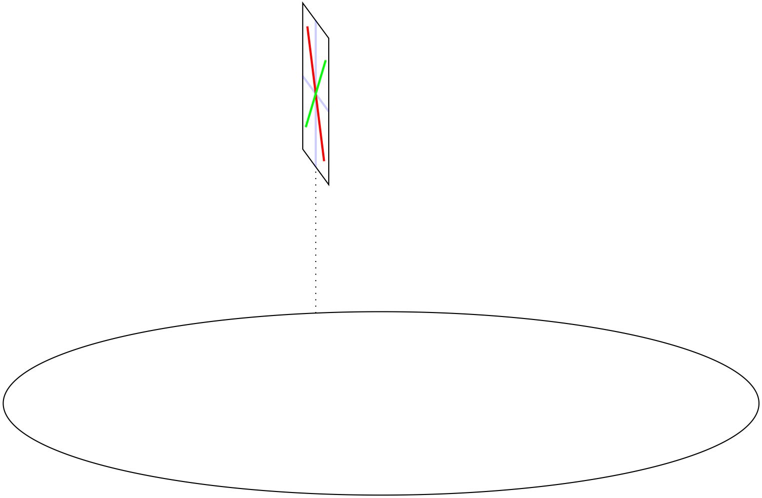

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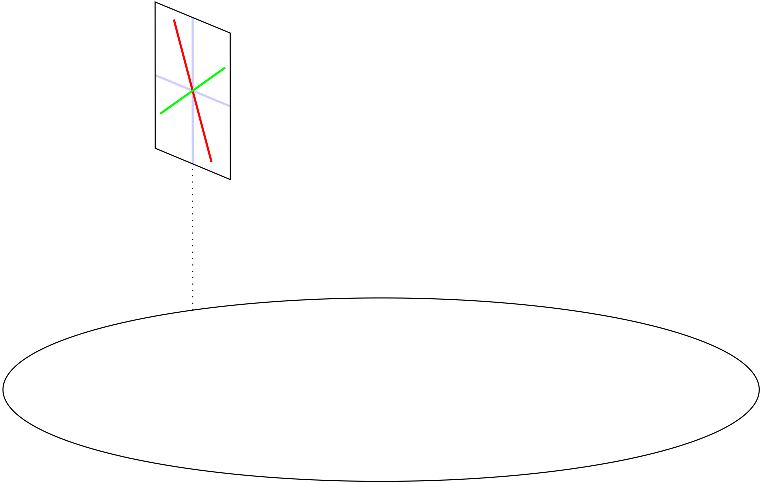

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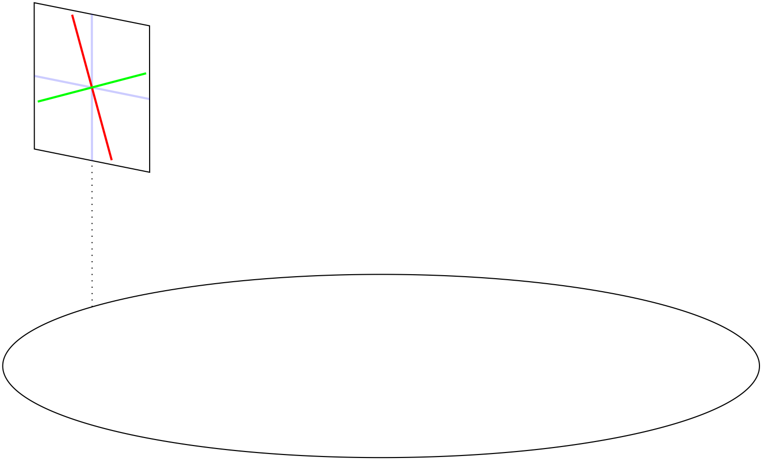

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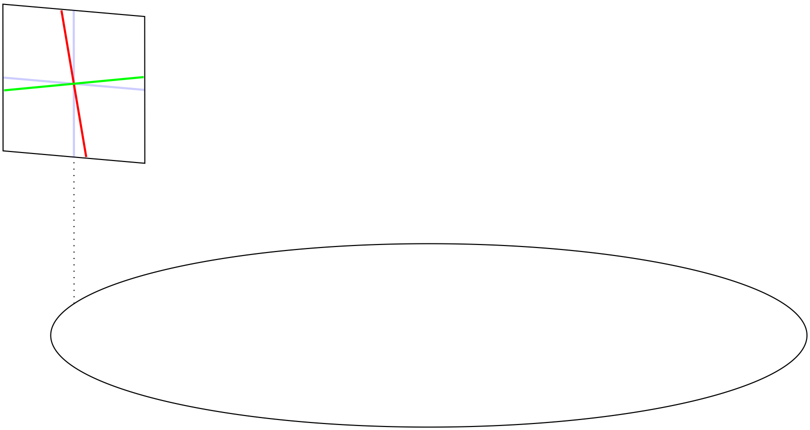

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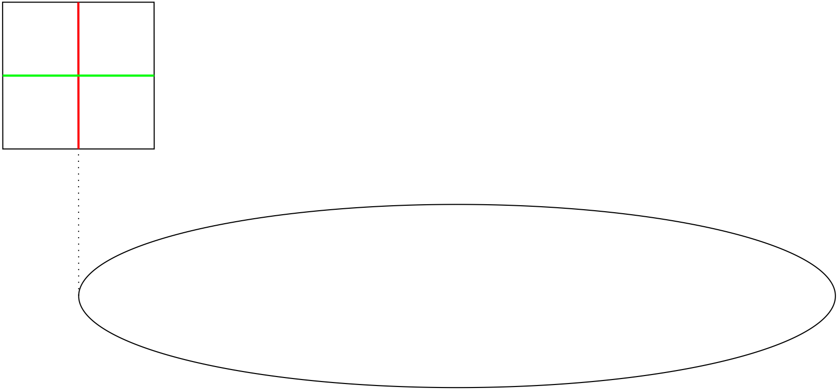

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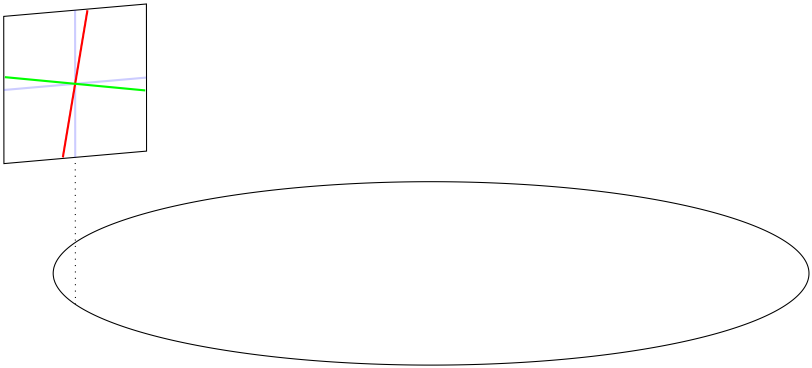

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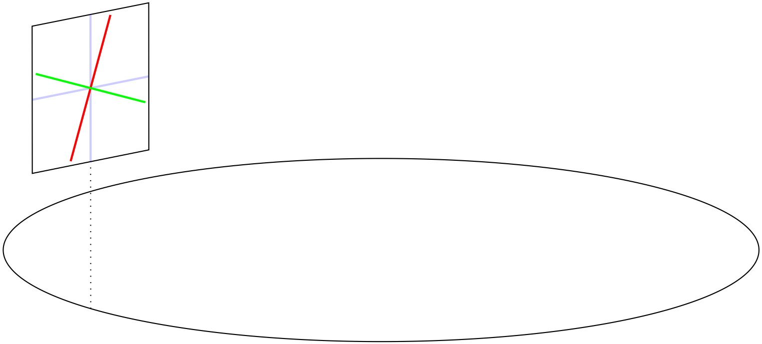

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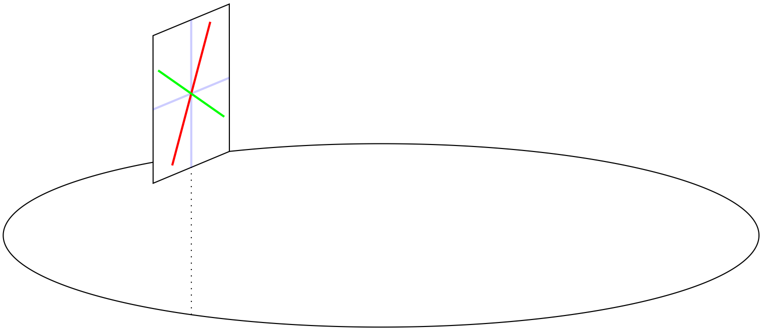

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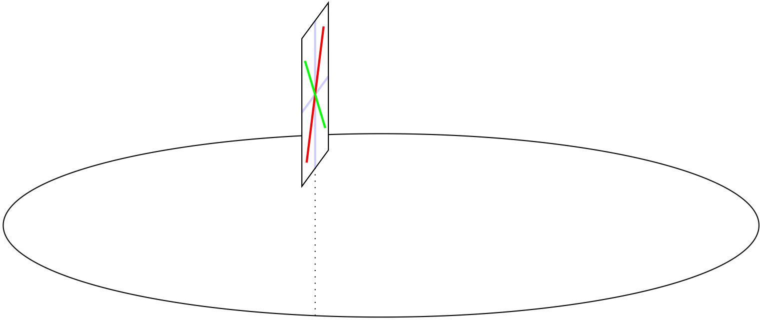

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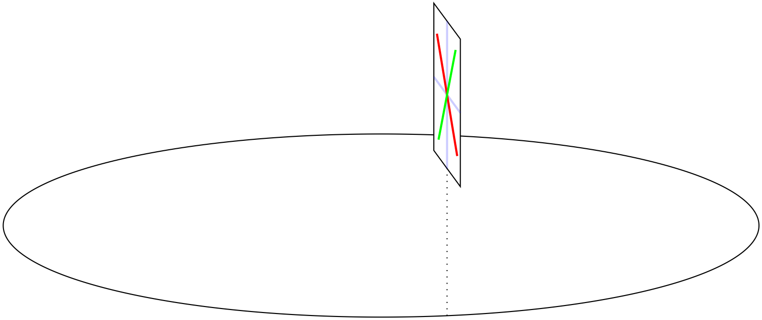

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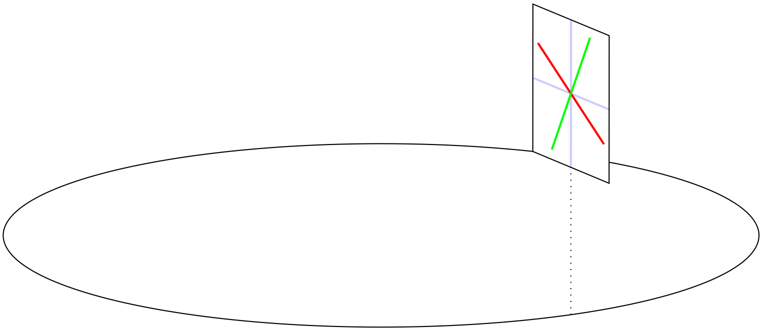

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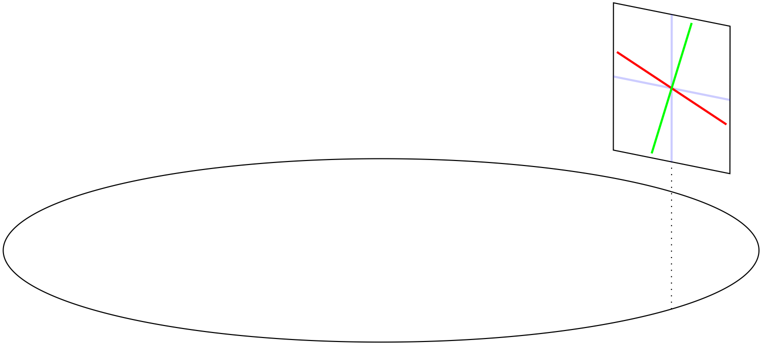

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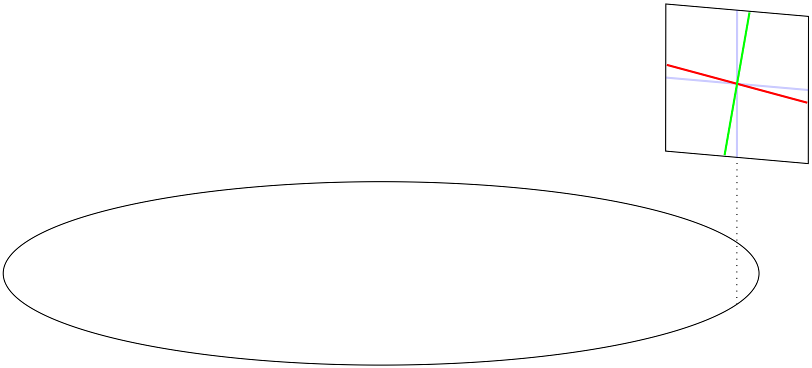

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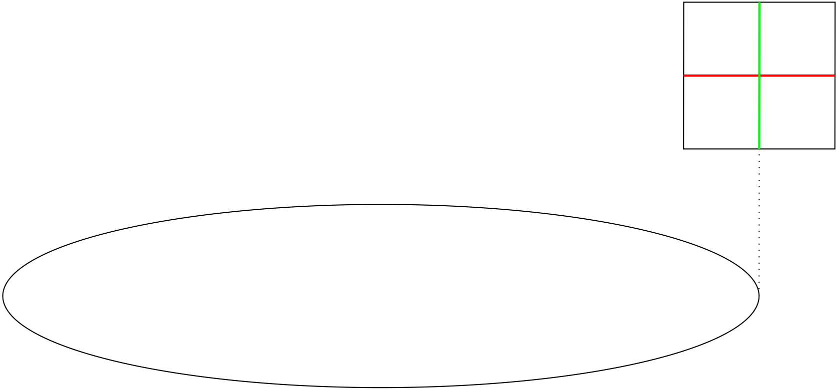


Vector bundles on the two dimensional sphere

- Understanding vector bundles on higher dimensional spheres is fun. See Allen Hatcher, "Vector bundles and K-theory", Section 1.2.
- Example: The list of indecomposable vector bundles on S^2 looks as follows

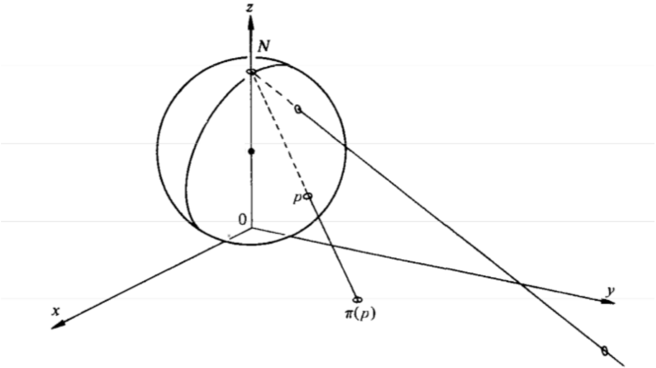
$$\underbrace{\mathcal{O}}_{\text{trivial line bundle}}, \overbrace{\underbrace{\mathcal{E}(1)}_{\substack{\text{analogue of}\\ \text{Möbius bundle}}}, \underbrace{\mathcal{E}(2)}_{\text{tangent bundle}}, \mathcal{E}(3), \ldots}^{\text{rank 2}}$$

- In rank $k \geq 3$ things collapse drastically and we only have

$$\underbrace{\mathcal{O} \oplus \cdots \oplus \mathcal{O}}_{k} \quad \text{and} \quad \underbrace{\mathcal{O} \oplus \cdots \oplus \mathcal{O}}_{k-2} \oplus \mathcal{E}(1)$$

Holomorphic vector bundles

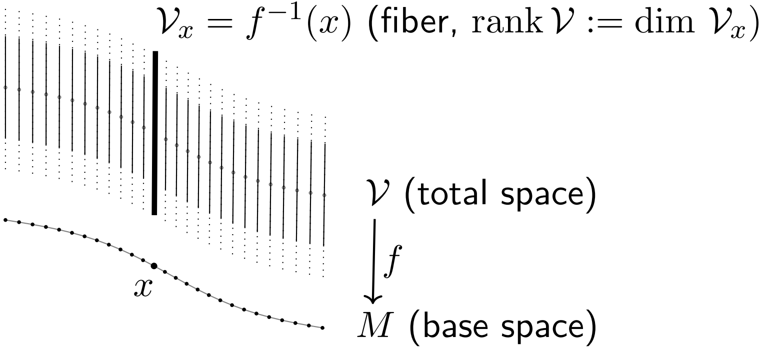
- Stereographic projection identifies S^2 with $\mathbb{R}^2 \cup \{\infty\} \cong \mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1(\mathbb{C})$.



- In this way S^2 becomes a complex manifold.

  Reminder: A complex manifold has charts with coordinates given by complex numbers and coordinate changes consisting of holomorphic functions.

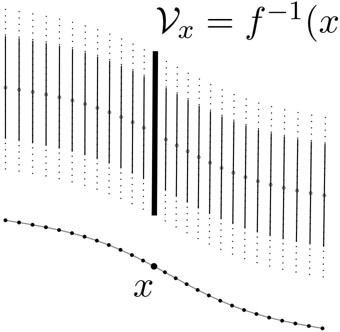
Holomorphic vector bundles



- If M is a complex manifold then it is natural to require that $\mathcal{V}$ is a complex manifold as well, and that f is a morphism of complex manifolds. Then we call $\mathcal{V}$ a holomorphic vector bundle.

Holomorphic vector bundles

$\mathcal{V}_x = f^{-1}(x)$ (fiber, $\operatorname{rank}\mathcal{V} := \dim\, \mathcal{V}_x$)



$$\begin{array}{c} \mathcal{V} \text{ (total space)} \\ \downarrow f \\ M \text{ (base space)} \end{array}$$

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- Example: the tangent bundle on a complex manifold (like S^2) is holomorphic.

Holomorphic vector bundles

- The holomorphic line bundles on S^2 are indexed by $\mathbb{Z}$.

$$\ldots, \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3), ...$$

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- They are related to the real rank two bundles via $\mathcal{O}(n)_{\mathbb{R}} \cong \mathcal{E}(|n|)$ (for $|n| \geq 1$)
- Every holomorphic vector bundle on the 2-sphere is a direct sum of line bundles in a unique way (no relations).

Morphisms between vector bundles

- A morphism $\phi : \mathcal{V} \to \mathcal{W}$ between vector bundles on M is a commutative diagram

$$\begin{tikzcd} \mathcal{V} \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{W} \arrow[dl] \\ & M & \end{tikzcd}$$

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- On the complex manifold S^2 we have

$$\dim \mathrm{Hom}_{S^2}(\mathcal{O}(m), \mathcal{O}(n)) = \begin{cases} n - m + 1 & \text{if } n \geq m \\ 0 & \text{otherwise} \end{cases}$$

Mirror symmetry for the sphere

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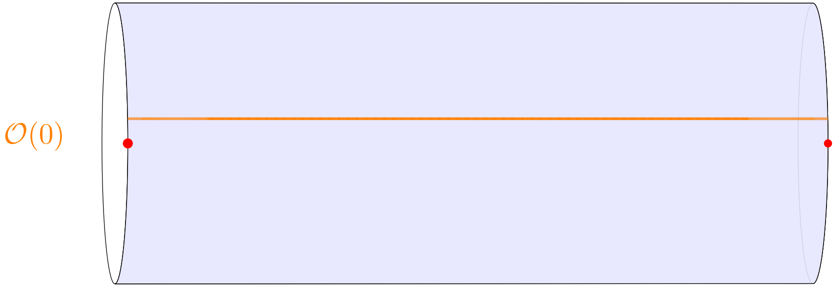

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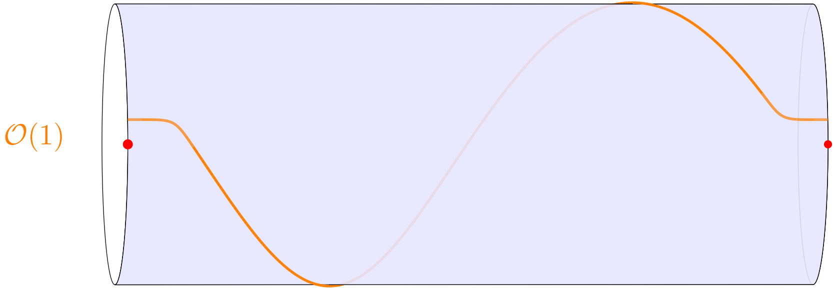
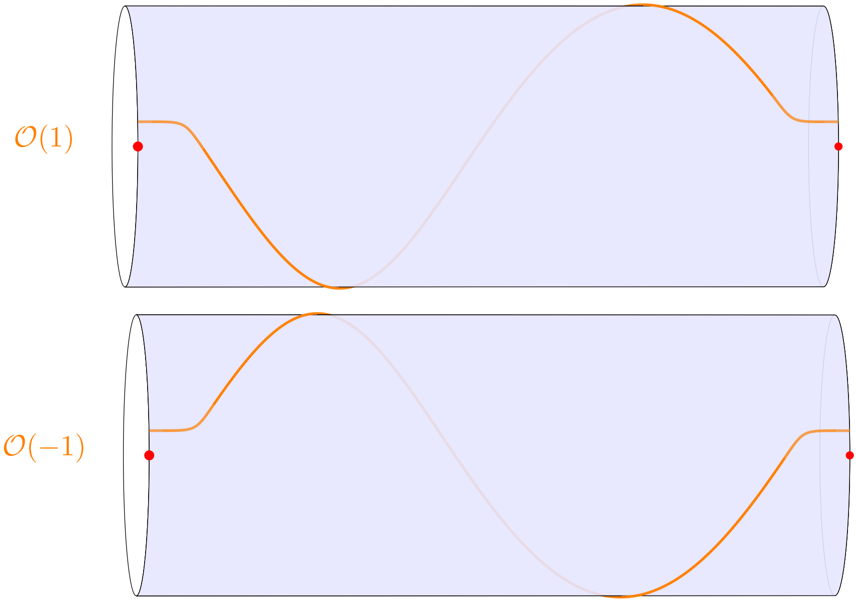


Holomorphic line bundles on S^2 as curves



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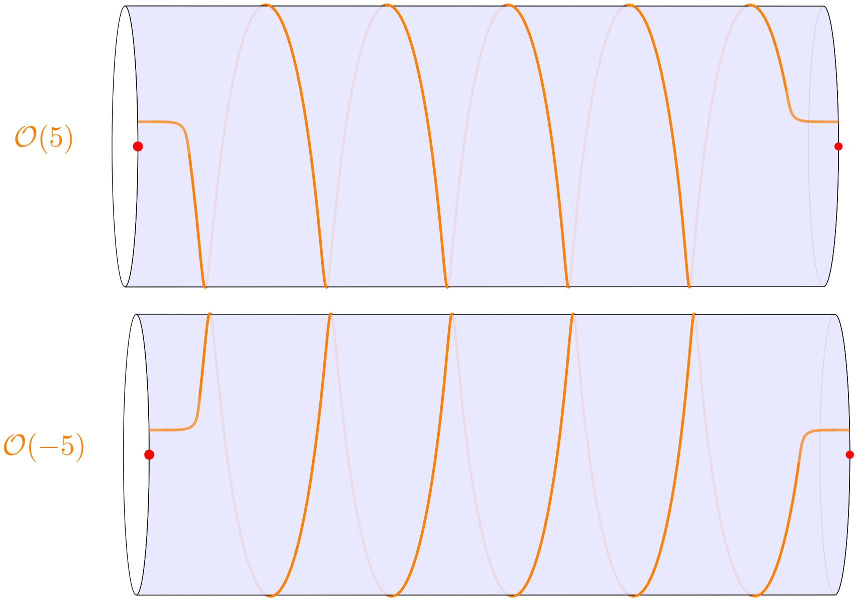
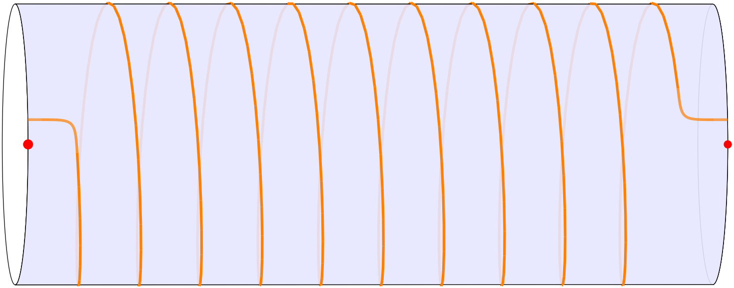


Holomorphic line bundles on S^2 as curves

$\mathcal{O}(10)$



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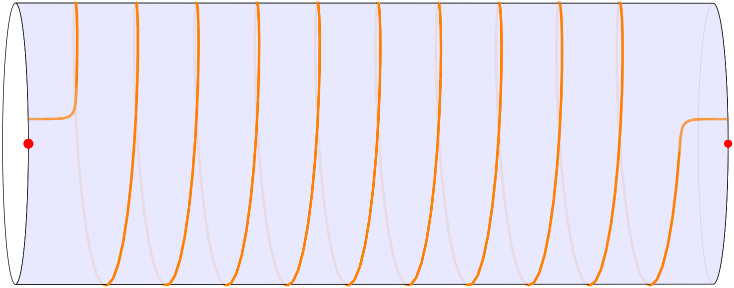
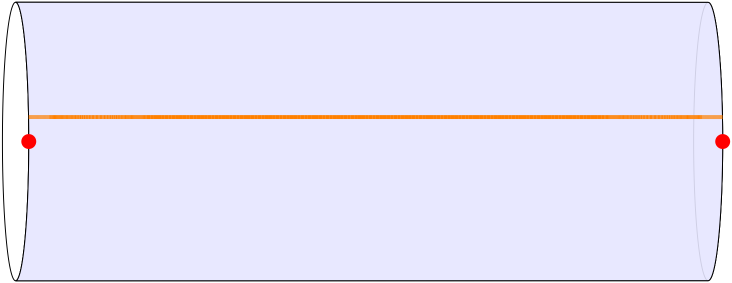


Wrapping

- If C_m is the curve corresponding to $\mathcal{O}(m)$ then very roughly speaking $\mathrm{Hom}(\mathcal{O}(m), \mathcal{O}(n))$ has a basis indexed by

$$\text{wrapping}(C_m) \cap C_n$$

where the wrapping operation is described by the following animation.



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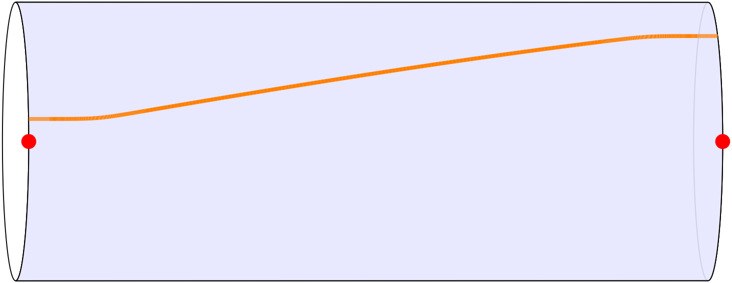

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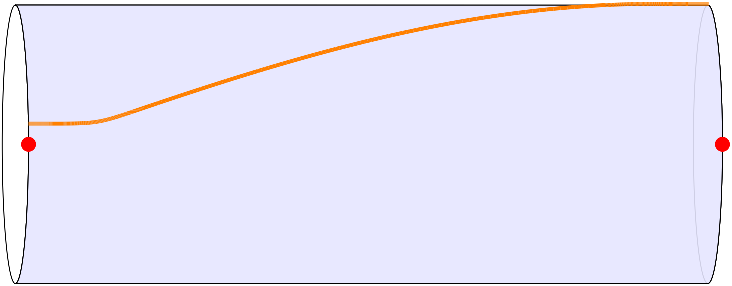

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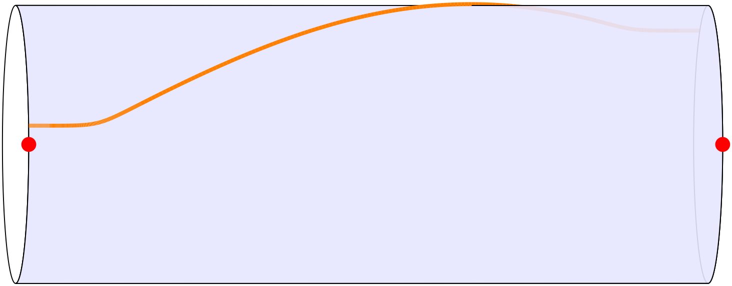

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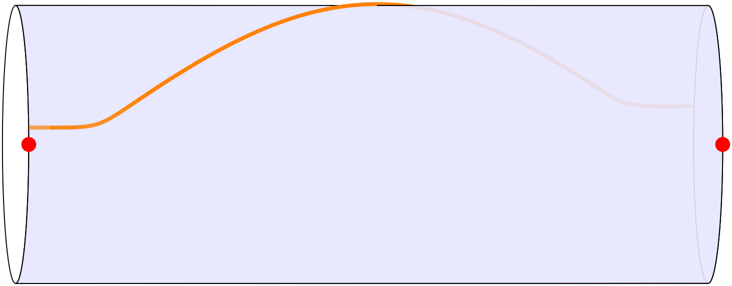

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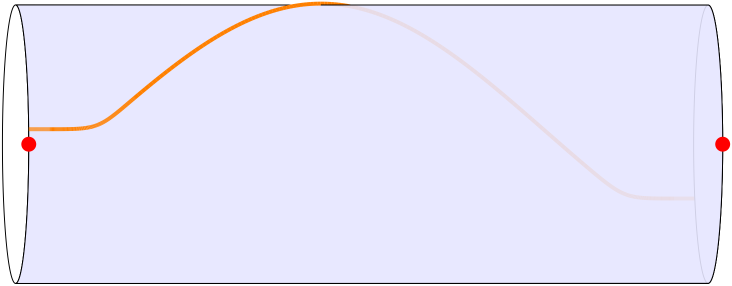
where the wrapping operation is described by the following animation.



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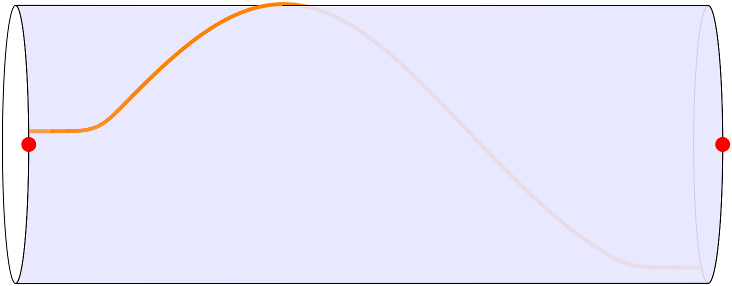

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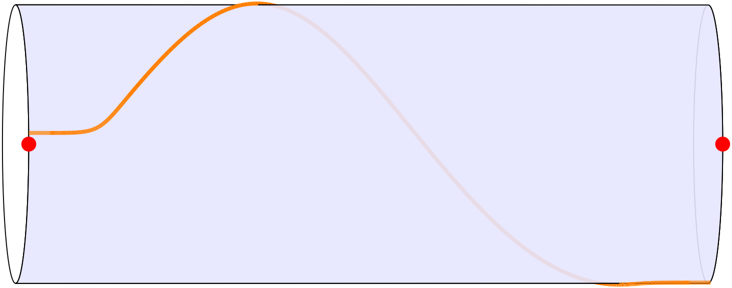

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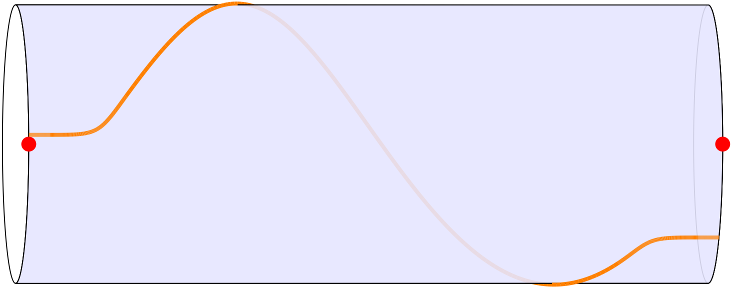

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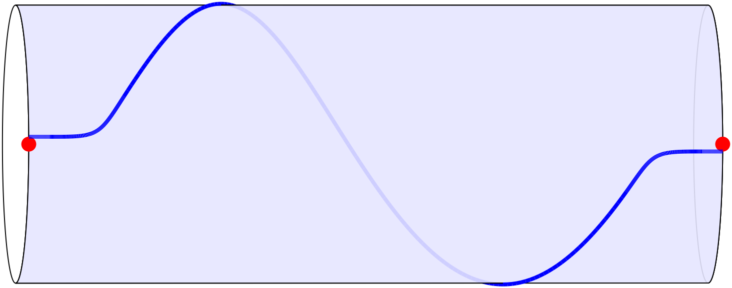
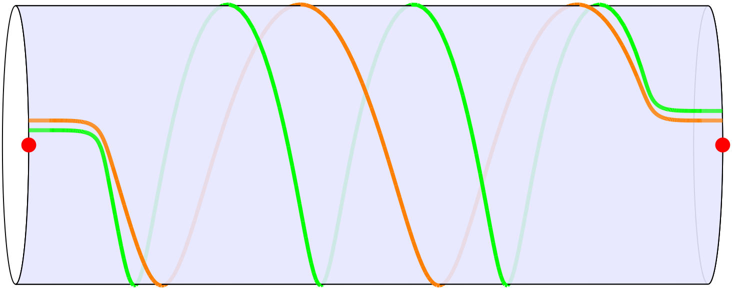


Done!

Example 1: $\dim \mathrm{Hom}(\mathcal{O}(2), \mathcal{O}(3)) = 2$



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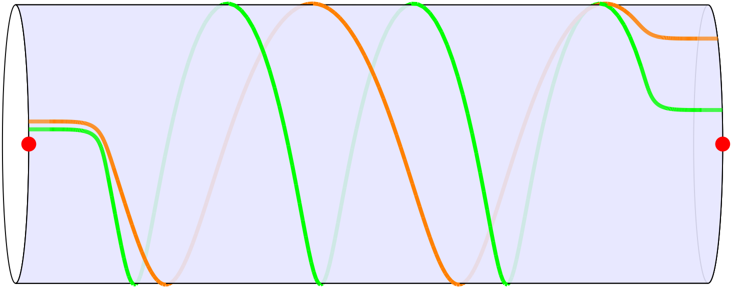

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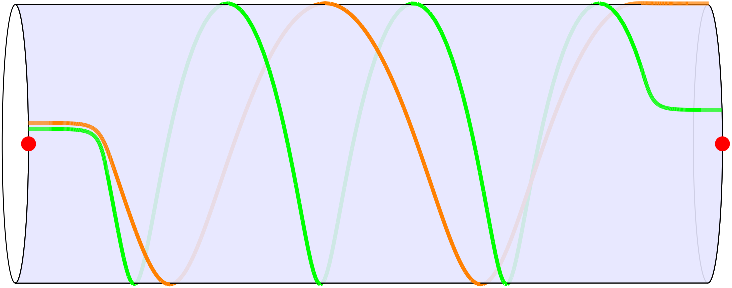

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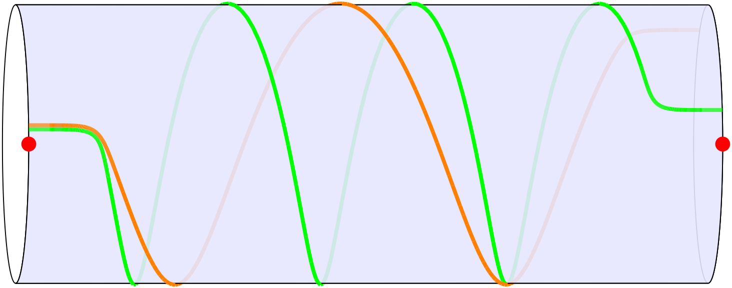

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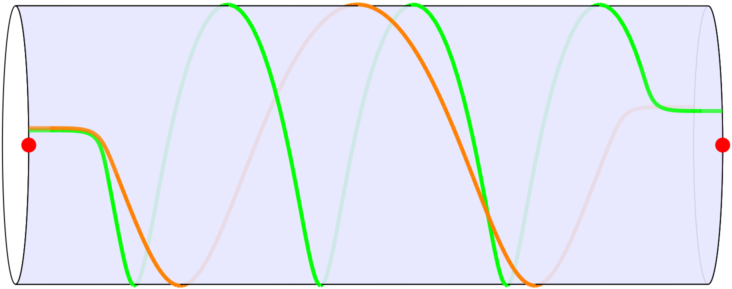

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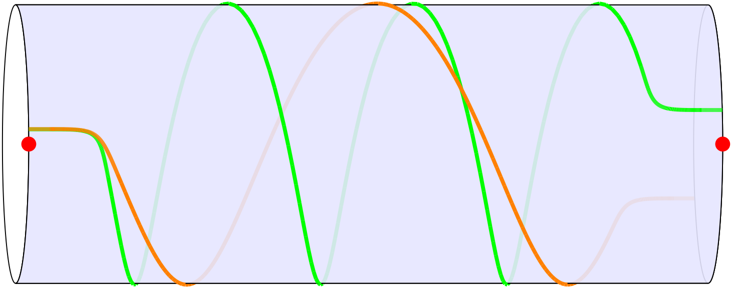

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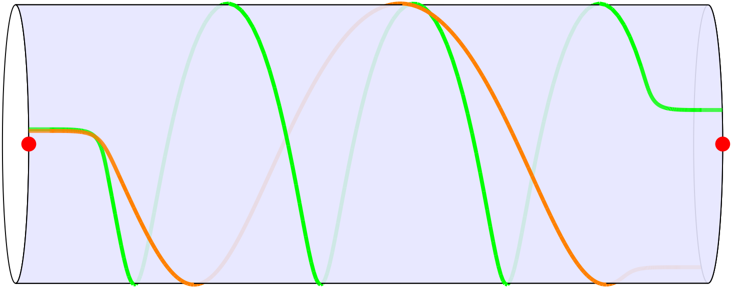

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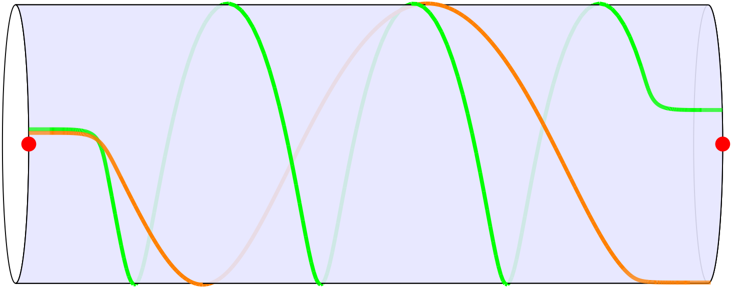

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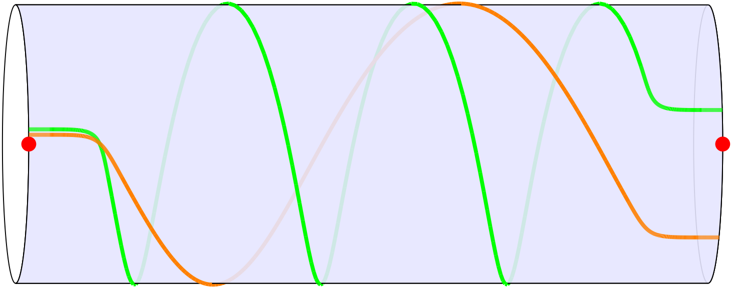

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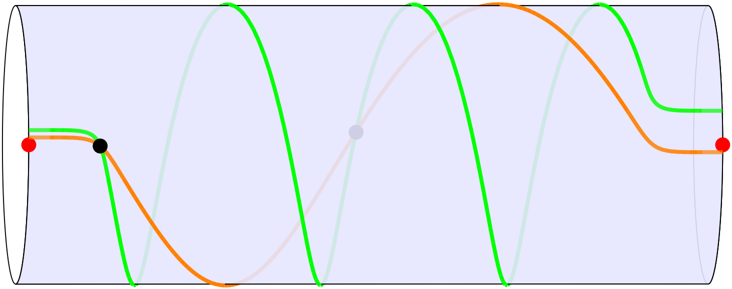
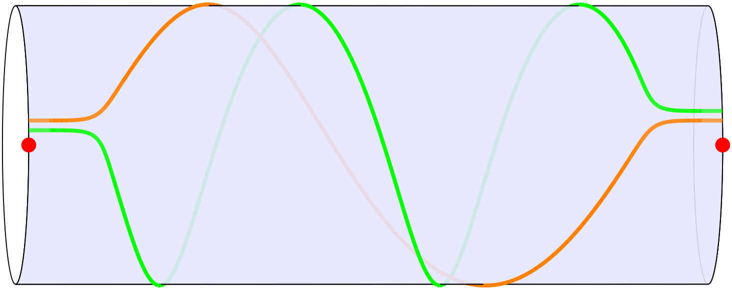


Finished

Example 2: $\dim \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}(2)) = 4$



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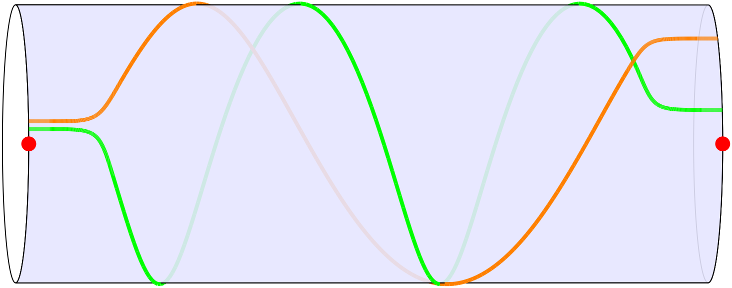

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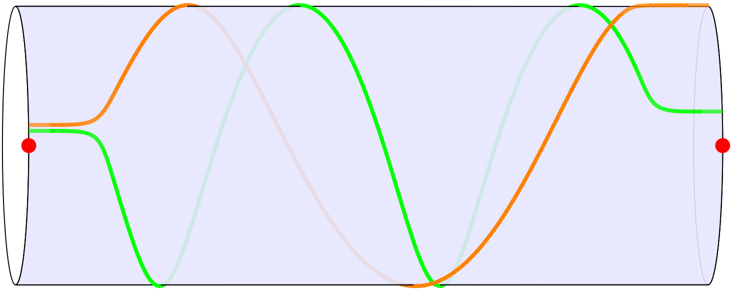

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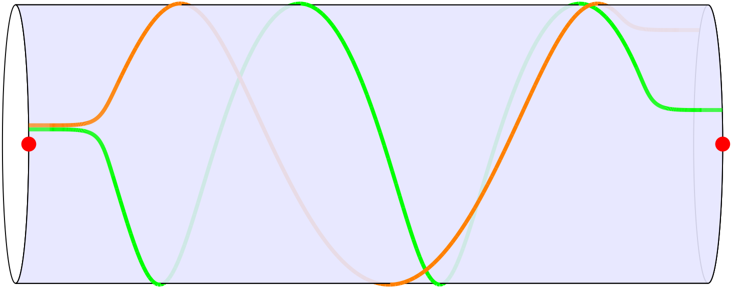

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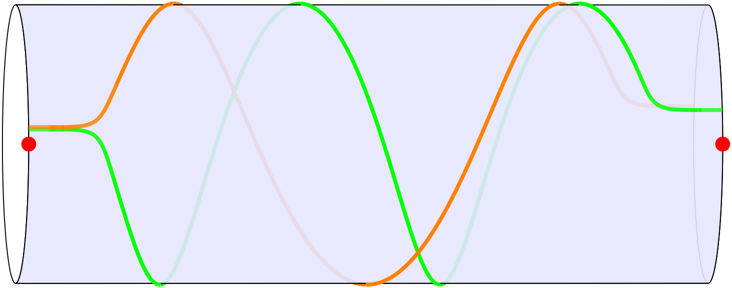

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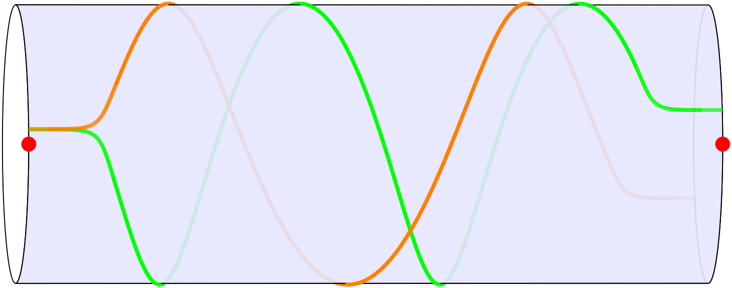

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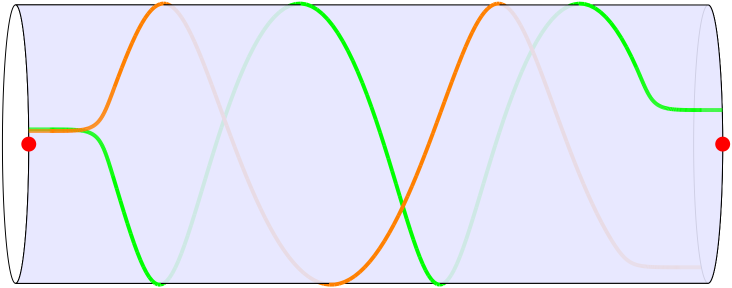

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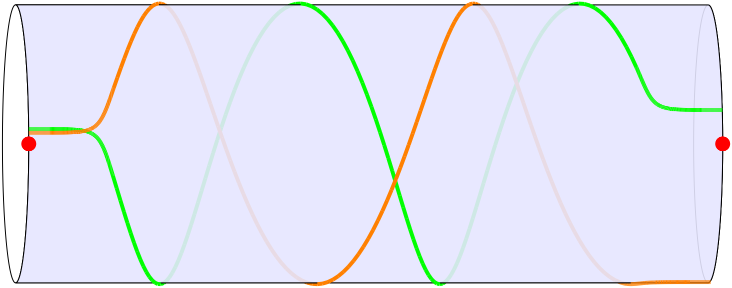

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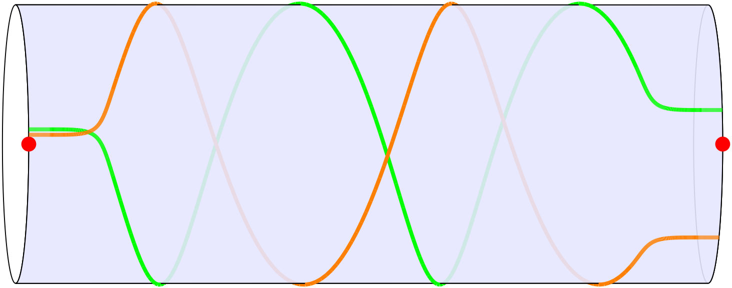

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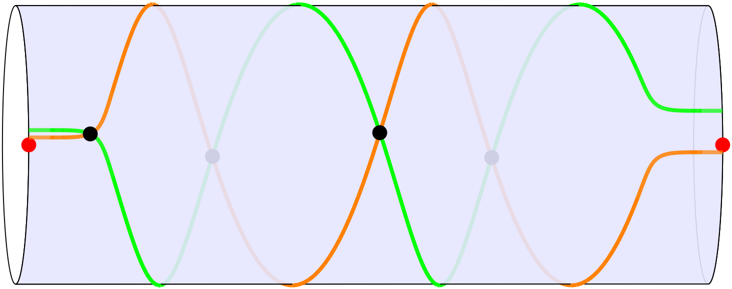
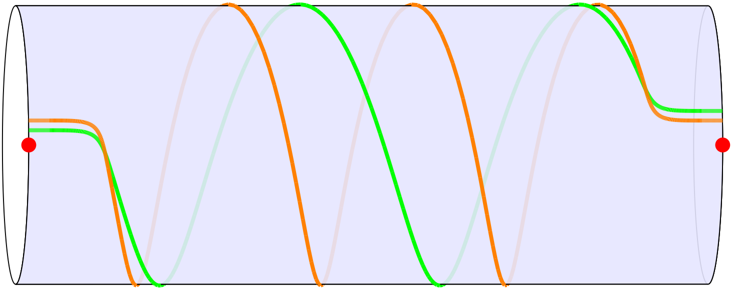


Finished

Example 3: $\dim \operatorname{Hom}(\mathcal{O}(3), \mathcal{O}(2)) = 0$



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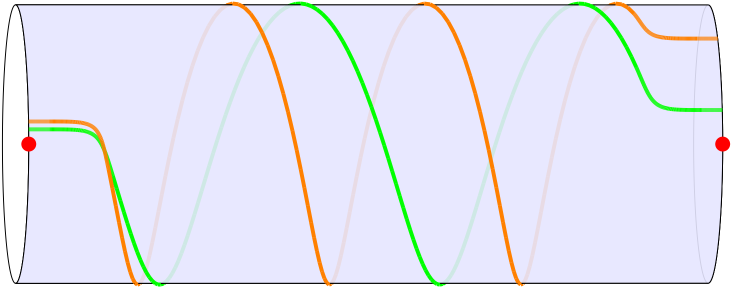

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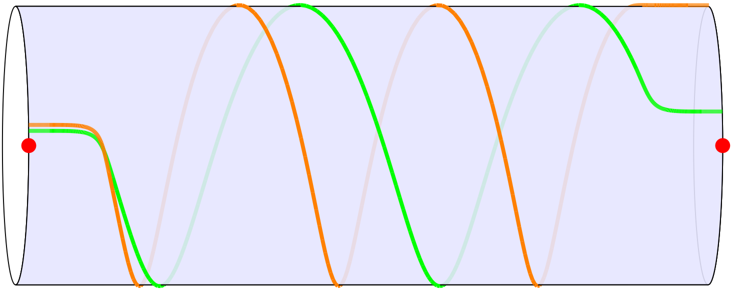

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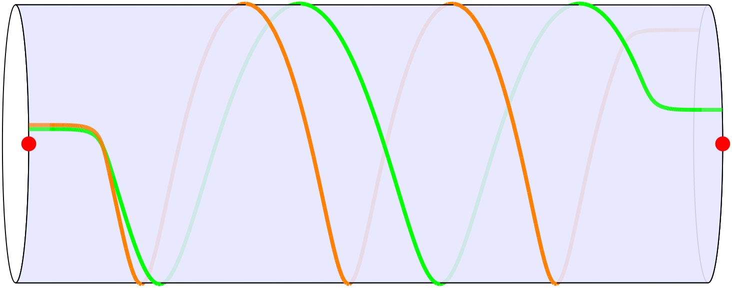

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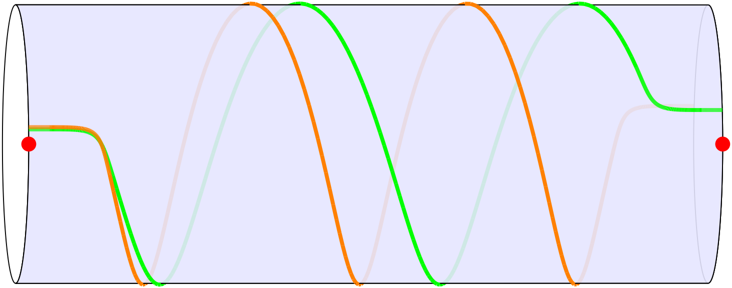

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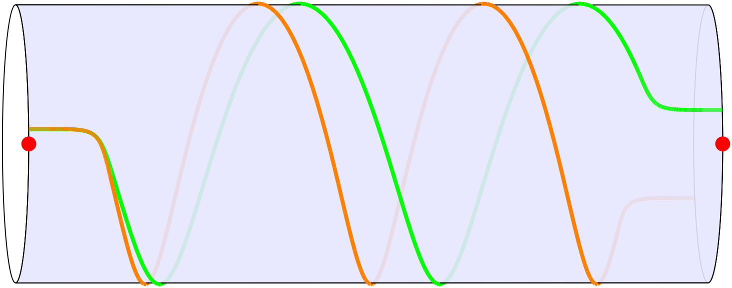

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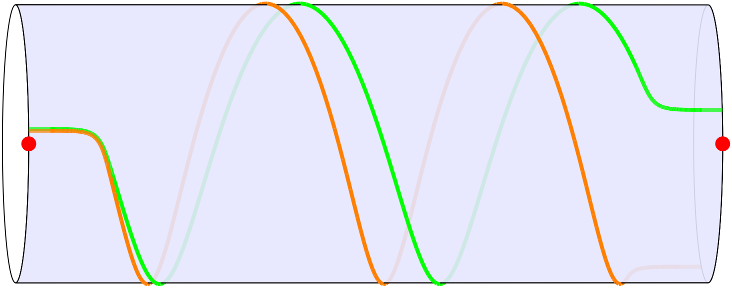

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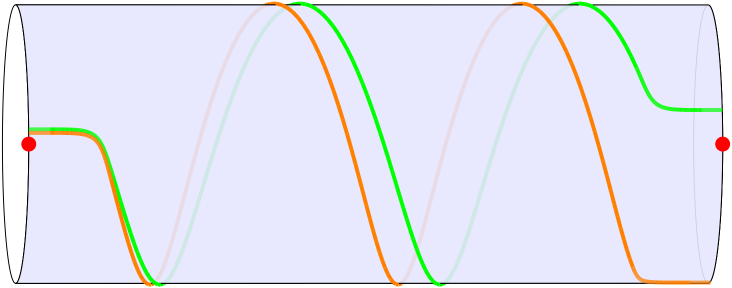

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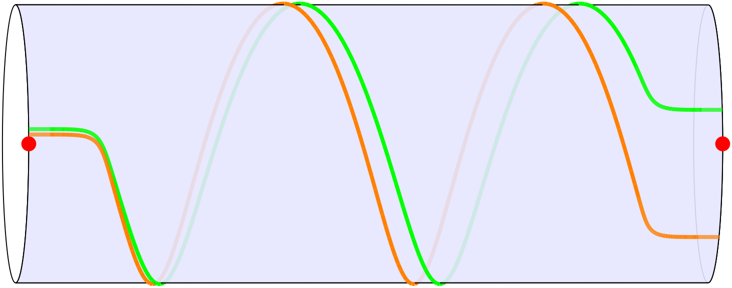

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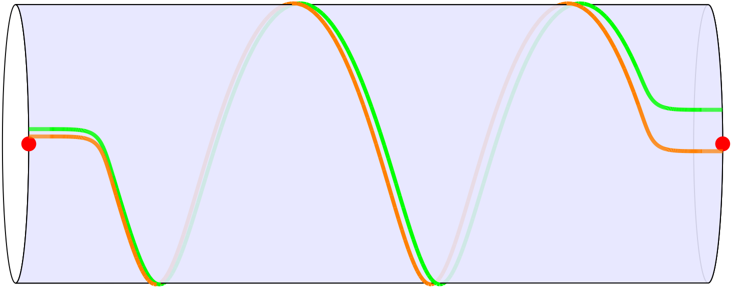

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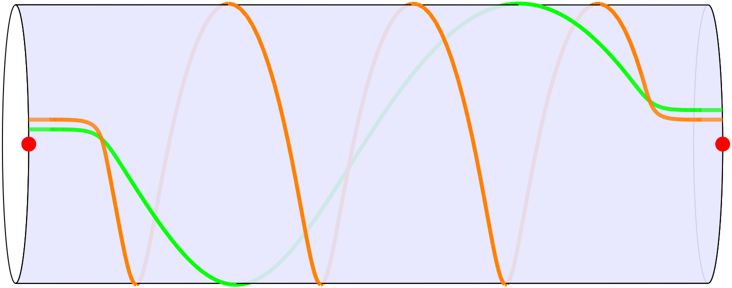

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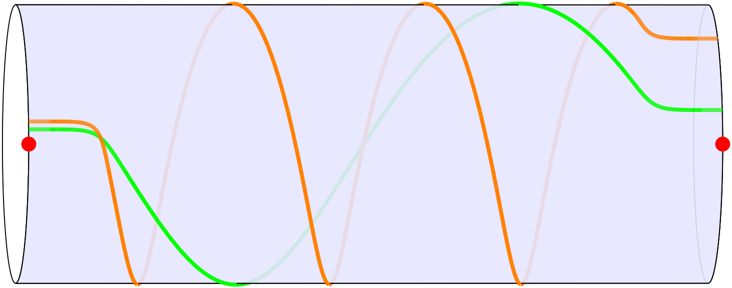

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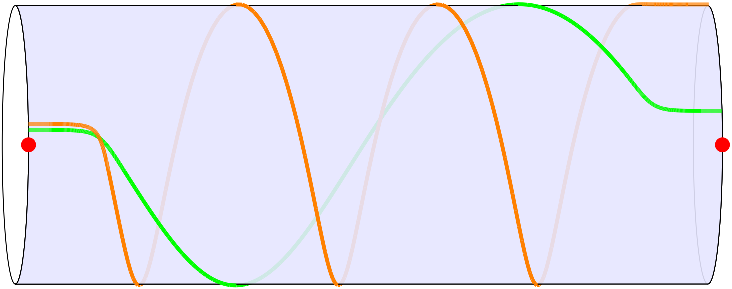

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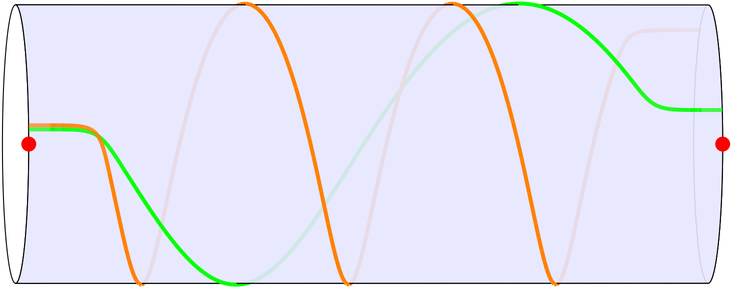

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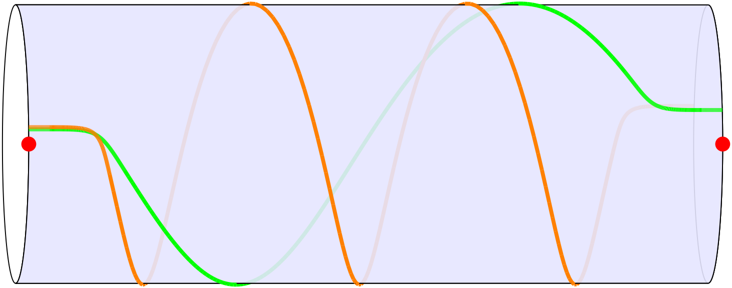

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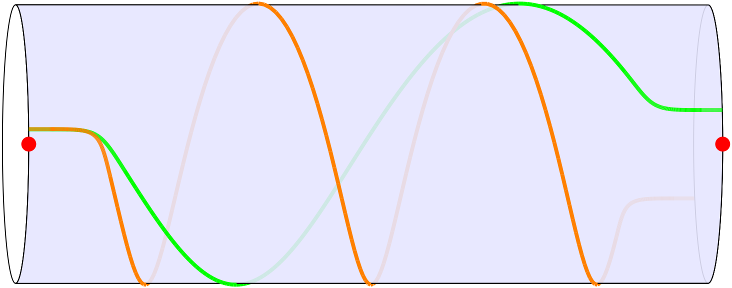

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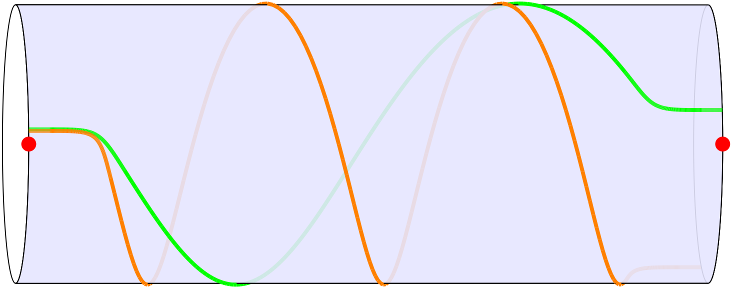

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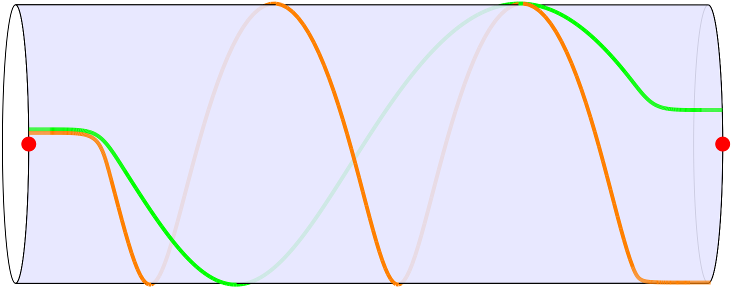

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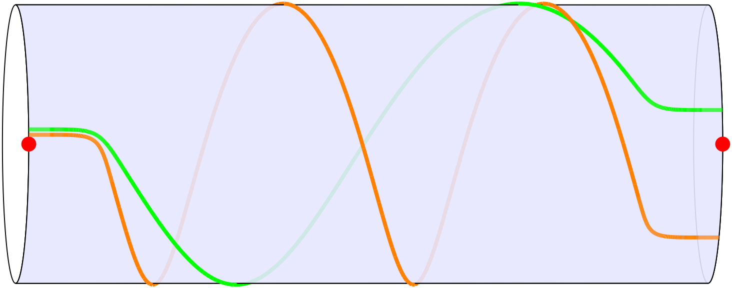

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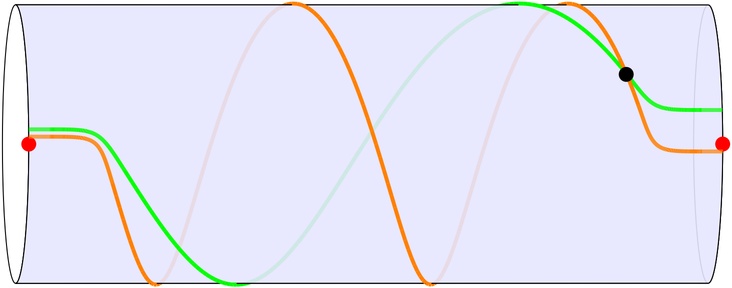

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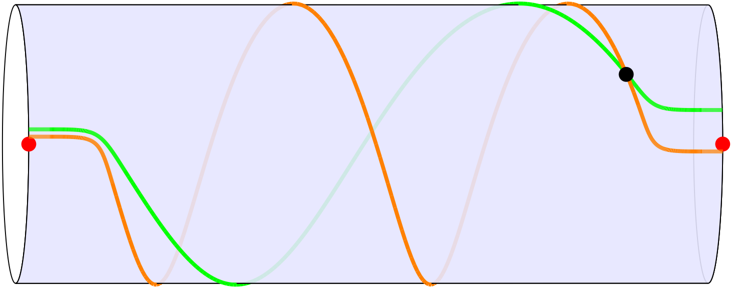

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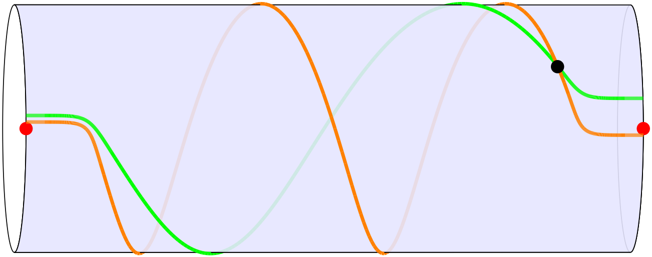

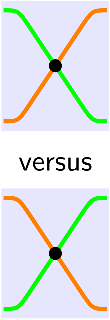

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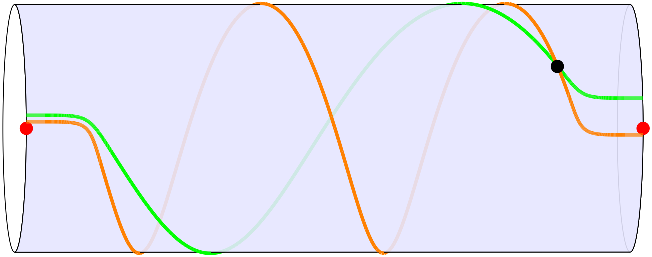

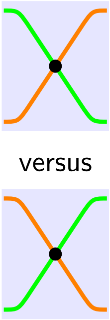

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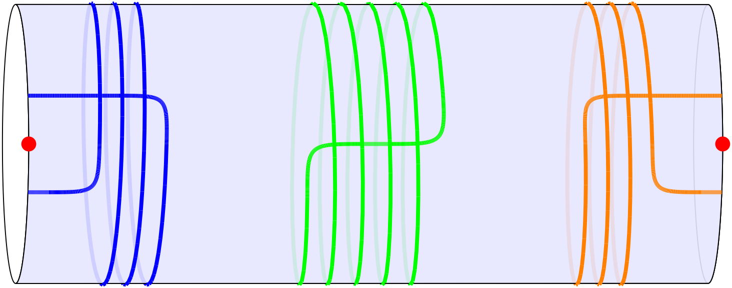
- It represents an element of a "higher Hom space" denoted by Ext^1.
- It turns out that in our case $\dim \mathrm{Ext}^1(\mathcal{O}(3), \mathcal{O}(1)) = 1$ ✔

Other curves

- There are other interesting non-contractible paths on the cylinder.



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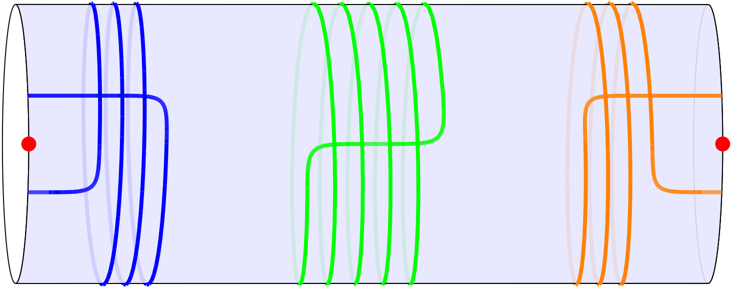

- These correspond to so-called "coherent sheaves" living on isolated points of S^2. Coherent sheaves are generalizations of vector bundles. Roughly speaking these have fibers of varying dimension (but one needs a better language to talk about them).

The SYZ principle

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$$\begin{tikzcd} X \arrow[dr, "f"'] & & \hat{X} \arrow[dl, "\hat{f}"] \\ & B & \end{tikzcd}$$

to a common base space B such that over a large subset of B, the fibers of f and $\hat{f}$ are "dual" tori (i.e. spaces of the form $S^1 \times \cdots \times S^1$).

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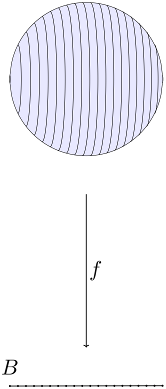
  to a common base space B such that over a large subset of B, the fibers of f and $\hat{f}$ are "dual" tori (i.e. spaces of the form $S^1 \times \cdots \times S^1$).
- Line bundles on X correspond to (Lagrangian) sections of $\hat{f}$.

Example: the 2-sphere



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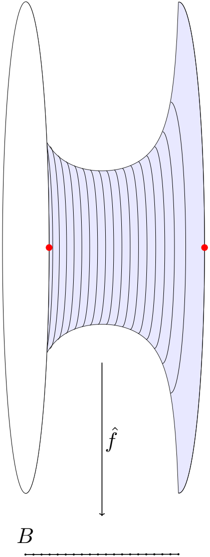
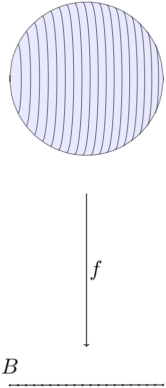
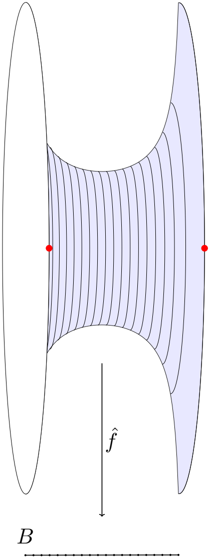


The SYZ mirror

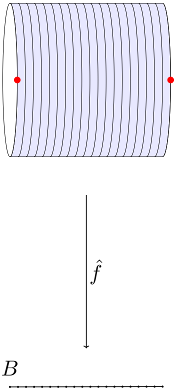
Example: the 2-sphere



The 2-sphere



The SYZ mirror



(simplified)

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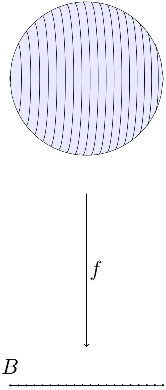

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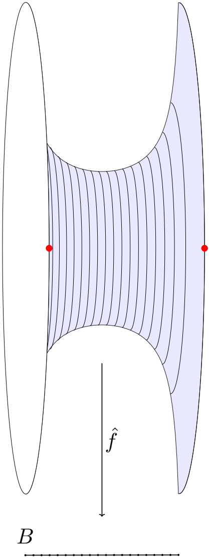

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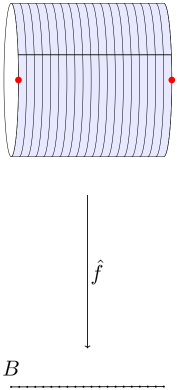
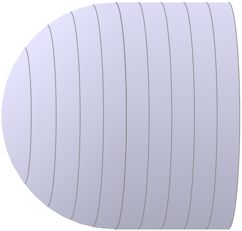


(section for $\mathcal{O}(0)$)

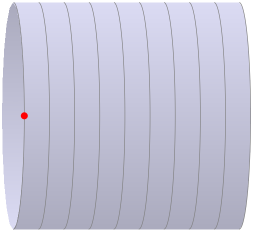
Other examples: the plane

- The mirror of the plane $\mathbb{R}^2 \cong \mathbb{C} \cong S^2 - \{\text{point}\}$ is a cylinder with one stop on the boundary.

- SYZ view

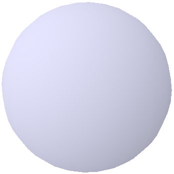


The plane

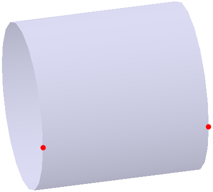


The mirror

Comparison with the sphere: “stop removal”

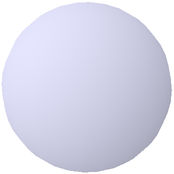


The sphere (S^2)



The mirror (2 stops)

Comparison with the sphere: “stop removal”



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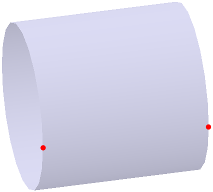

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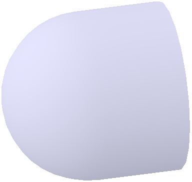
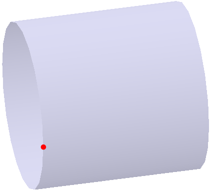
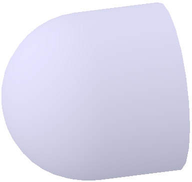


The plane ($S^2 - \{\text{point}\}$)

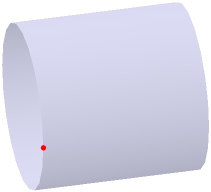


The mirror (1 stop)

Removing the last stop



The plane ($S^2 - \{\text{point}\}$)



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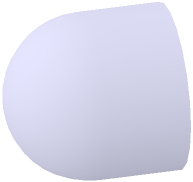
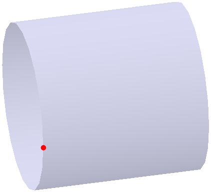


The plane ($S^2 - \{\text{point}\}$)



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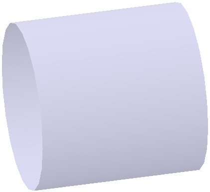
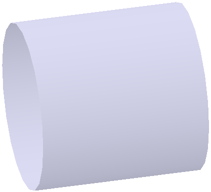


The cylinder ($S^2 - \{2 \text{ points}\}$)



The mirror (0 stops)

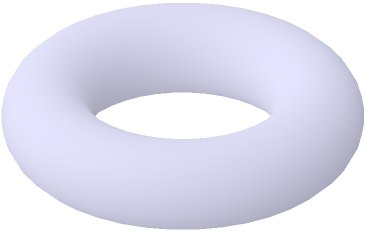
The mirror of a cylinder is a cylinder...

Other examples: the torus

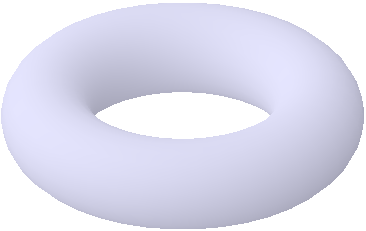
For $\lambda \in \mathbb{C} - \{0, 1\}$ we obtain a complex torus via

$$\bar{E}_\lambda = \{(X, Y, Z) \in \mathbb{P}^2(\mathbb{C}) \mid Y^2 Z = X(X - Z)(X - \lambda Z)\}$$

Just like a cylinder, the mirror of a torus is a torus



A torus

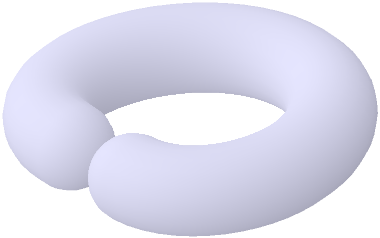


The mirror

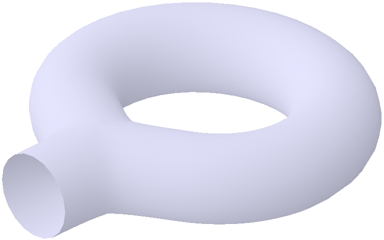
Other examples: the pinched torus

For $\lambda \in \{0, 1\}$ the torus $\bar{E}_\lambda$ becomes pinched.

The mirror of a pinched torus is a punctured torus...



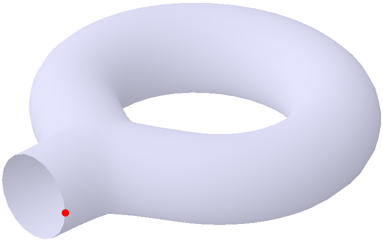
A pinched torus



The mirror

Putting a stop on the puncture

- Question: what happens on the complex side if we put a stop on a punctured torus on the symplectic side?



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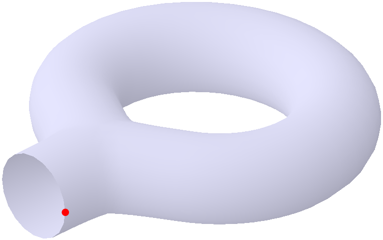

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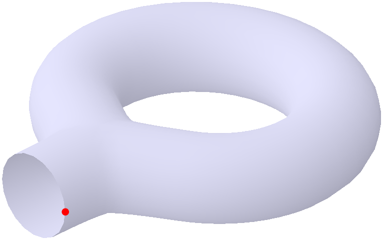

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- Answer: we get a noncommutative space on the complex side (Lekili-Polishchuk)...

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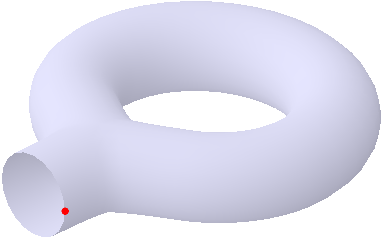
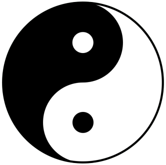


- By the inverse of stop removal we would have to make the complex side bigger. But the complex side is already "compact".
- Answer: we get a noncommutative space on the complex side (Lekili-Polishchuk)...
- A new kind of geometry: noncommutative geometry...

Thank you



Some references: Abouzaid, Auroux, Batyrev, Bocklandt, Burban, Drozd, Dyckerhoff, Ganatra, Haiden, Kapranov, Katzarkov, Kontsevich, Lekili, Pardon, Polishchuk, Seidel, Shende, Zaslow, ...